Hans-Joachim Petsche

Hermann Graßmann

Biography

Translated by Mark Minnes

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Translator's note

Prof. Hans-Joachim Petsche's *Graßmann* is a book on mathematics and German cultural history. As a translator, I have attempted to make the text as accessible as possible to the English-speaking readership. Some decisions should not go unmentioned.

Hermann Graßmann's mathematical terminology is often quite extravagant and unusual. In many cases, the translation also gives Graßmann's original German concepts. With his permission and generous cooperation, I have used Dr. Lloyd Kannenberg's terminology from his translations of Graßmann, e. g. displacement ("Strecke"), magnitude ("Größe"), conjunction ("Verknüpfung"), evolution ("Änderung"), etc.

As many German-language books in the original bibliography as possible have been cited from their English translations. In many cases, however, quotations had to be translated and the footnotes still refer to the German originals. The Graßmann brothers used an early edition of Schleiermacher's *Dialectic* of which there is no English translation. Of course, the same is true of many other works and letters.

All titles of books and journal articles have been translated into English. The original titles and the corresponding bibliographical references appear in parentheses and quotation marks. In these cases, dates refer to the year of publication of the text quoted in the bibliography.

Finally, we should not forget that today the formerly Prussian town of Stettin is the Polish city of Szczecin. The use of the German name Stettin should not be construed as questioning that historical fact.

This translation was a collaborative process. I would like to thank Dr. Steve Russ (University of Warwick) and Dr. Lloyd Kannenberg (University of Massachusetts Lowell) for sharing their expertise and doing the hard work of proofreading the manuscript. I thank them for many pleasant discussions and their supportive approach to the project. The author, Hans-Joachim Petsche, kept our little research team together and showed warm and stimulating appreciation of our efforts.

Mark Minnes

Foreword

In Wilhelm Traugott Krug's *General Handbook of the Philosophical Sciences* of 1827 ("Allgemeines Handwörterbuch der philosophischen Wissenschaften"), we find the following entry for the terms "mathematics" and "mathematical":

"Mathematics ... only [deals with] magnitudes which appear in time and space and which therefore can be represented, counted and measured as numbers or figures... A philosopher should familiarize himself with mathematics and a mathematician with philosophy, as far as their talent, interests, time and surroundings will permit. But one should not confuse and throw into one pot what the progress of scientific knowledge has separated, and rightly so. ...mathematical philosophy and philosophical mathematics – in the commonly accepted sense of the terms, namely as a mixture of both – are scientific or, rather, unscientific monsters. They no more satisfy and please the educated mind than could a human body consisting of a mixture of man and woman."[1]

But this view did not prevent Hermann Graßmann[2], a 35 year-old secondary school teacher from the Prussian town of Stettin, from publishing a work of mathematics which, as he later remarked, "is certain to be more pleasing to more philosophically inclined readers".[3] Graßmann's book also claimed to have founded a new branch of science "which extends and intellectualizes the sensual intuitions of geometry into general, logical concepts, and, with regard to abstract generality, is not simply one among the other branches of mathematics, such as algebra, combination theory, and function theory, but rather far surpasses them, in that all fundamental elements are unified under this branch, which thus as it were forms the keystone of the entire structure of mathematics."[4]

Hermann Graßmann, a novice in mathematics whose name was completely unknown in the mathematical community of his day, did not hesitate to send his book – *Linear Extension Theory, A New Branch of Mathematics* (1844) – to the most famous mathematicians of his time. But their assessment of his work remained completely within the framework of the Kantian view of mathematics, which we find in the quotation above. This was a disaster for Graßmann. Among German mathematicians, August Ferdinand Möbius was closest to Graßmann's scientific perspective. Möbius told Apelt in a letter that he had repeatedly attempted to understand Graßmann's book, "...but I never got beyond the first pages ... since [the book] ... lacks all intuitive clarity, which is the essential characteristic of mathematical insight."[5] In a letter to Gauß, Möbius wrote that Graßmann had "strayed from the firm foundations of mathematics"[6]. Johann August Grunert wrote Graßmann: "I also would have hoped that you would have refrained from getting so involved in philosophical reflections."[7] Ernst Friedrich Apelt, a friend of Möbius, remarked that "Graßmann's peculiar *Extension Theory* ... seems to be built on a wrong understanding of the philosophy of mathematics. ...The *abstract* extension theory he is looking for could only be developed from concepts. Concepts are not the source of mathematical knowledge, but intuition."[8] Finally, Richard Baltzer came to the following conclusion: "...I begin to feel dizzy in the head and disoriented when I read it."[9] Moritz Cantor summed up the fate of *Extension Theory* in one simple sentence: "The book was published in 1844 by O. Wigand in Leipzig, nobody reviewed it, nobody bought it, and therefore the publisher destroyed the entire first edition!"[10] Half a century later, nobody doubted the importance of Graßmann's mathematical work. On Felix Klein's initiative, a six-volume collection of Graßmann's writings in mathematics and physics was published between 1894 and 1911.[11] Thanks to mathematicians such as Hermann Hankel, Alfred Clebsch, Felix Klein and Friedrich Engel, Graßmann's achievements concerning the foundations of vector and tensor calculus, the development of n-dimensional affine and projective geometry and his fundamental work in algebra and in other areas were recognized in retrospect. Today, Graßmann has become a familiar name in mathematics. Nevertheless, many mathematicians are quite unfamiliar with his *magnum opus* in mathematics. Even though "a general feeling of respect for this mathematician from Stettin has spread in the scientific community", as F. Engel remarked in 1911, "this feeling of respect usually does not arise from knowledge of *Graßmann's* writings but, rather, is based on hearsay."[12] Graßmann's *Extension Theory* of 1844 was ignored for over a quarter of a century. Among other reasons, the general rejection of its philosophical approach and its philosophical mode of presentation led to this lack of recognition. Unhappily, this anti-philosophical attitude blinded mathematicians to the true value of *Extension Theory*. A closer analysis of the book will show that Graßmann's philosophical and, to put it more precisely, *dialectical* approach to mathematical problems is exactly what gave him the inspiration he needed to create and elaborate a new mathematical discipline, namely vector and tensor calculus. What is more: Graßmann was capable of building an unheard-of vector-algebraic theory of n dimensions because he was familiar with the philosophical thinking of his time and because he consciously used dialectics, the philosophy of the increasingly dominant German bourgeoisie, as a method for establishing and presenting new insights.

The present book aims to critically appreciate and explain the life and work of Hermann Graßmann (1809 – 1877).

Notwithstanding the fact that Graßmann has entered into the history of mathematics as the founder of vector algebra, he still remains a relatively unknown figure. The hundredth anniversary of his death in 1977 passed almost completely unnoticed. A conference held in Germany on the occasion of the 150th anniversary of the publication of *Linear Extension Theory* in May 1994 was one of the last major attempts to save his name from oblivion. In September 2009 the Graßmann Bicentennial Conference in Potsdam will commemorate the 200th anniversary of Graßmann's birth and attempt to contextualize his work from a present-day perspective.

The 19th century, in which Graßmann's scientific creativity blossomed, is still a highly promising area for future research. Few scholars have attempted to analyze the historical interactions between philosophy and mathematics. Very much remains to be done.[13]

These are plenty of reasons to have another look at Graßmann.

Introduction

Hermann Graßmann, born 200 years before the publication of the English edition of this book, was one of the most extraordinary personalities in 19th century science. The circumstances of his scientific achievements are no less remarkable than the results to which he came. Graßmann, who had originally aspired to become a theologian and remained an autodidact in mathematics and the natural sciences, was over 30 years old when he turned to scientific research. With the exception of three years in Berlin as a student, he spent virtually his entire life within the walls of his Pomeranian hometown, Stettin, where he worked as a teacher in a "Gymnasium", or secondary school. He had literally no contact to the leading scientists and mathematicians of his time and lived far away from the most important centers of scientific research. His personal library contained only a few scientific works. Nevertheless, he was extraordinarily prolific in his scientific work. Graßmann has gone down in history for his discoveries in the theory of electricity, the theory of colors and of vowels. He was also among the pioneers of comparative philology and of Vedaic research. In 1996, his dictionary of the vocabulary of the *Rig-Veda*, a collection of pre-Buddhist religious hymns from India (12th – 6th century BC), was reprinted for the sixth time.[14] Nonetheless, Graßmann's main achievements lie in the field of mathematics. His two "Ausdehnungslehren", or *Extension Theories* (A1, A2), of 1844 and 1861 make him one of the founders of vector and tensor calculus. Remarkably, he made these discoveries without having any connection to the English mathematician W. R. Hamilton. A decade before B. Riemann, Graßmann was the first mathematician to create a theory of *n*-dimensional manifolds by generalizing traditional three-dimensional geometry. Even though the results of his mathematical projects were not officially recognized for almost 30 years, they had an enduring scientific impact on mathematicians such as Felix Klein, Giuseppe Peano, Alfred North Whitehead, Élie Cartan, Hermann Hankel, Walter von Dyck, Josiah

Willard Gibbs, to name just a few. These facts alone should suffice to show that Graßmann's scientific achievement is well worth analyzing from the perspective of the history of science.

The present book is not the first to discuss Graßmann's life and work. Just one year after Graßmann's death, Victor Schlegel (1878) published a biography and, in 1911, when the six volumes of Graßmann's collected works appeared in Germany, Friedrich Engel published an extensive account of Graßmann's life (BIO). These two books are invaluable repositories of otherwise inaccessible documents concerning Graßmann's life. Yet, as scientific contributions aiming to establish Graßmann's place and relevance in the history of science, they show many flaws. When, for example, Schlegel claimed that Graßmann's mathematical conceptions arose "without the slightest connection to the historical development of science"[15], this expressed an extremely limited perspective on the history of mathematics. The only good thing we could say about this view is that it presumably was meant to make Graßmann's scientific achievement seem even greater.

The limitations of Schlegel's and Engel's views on the philosophical aspects of mathematics, which appear in their other works as well, also affect the biographies mentioned above. While Schlegel's biography rests on eulogistic judgments which are not open to an objective analysis of Graßmann's place in the history of mathematics, Engel's biography gives us a painstakingly precise portrayal of Graßmann's influence, but he abstains from making any judgment at all.

In contrast, from the perspective of the history of science, the present book aims to find out whether – and to what extent, and in which sense – the results of Graßmann's individual scientific brilliance, which seem to be the isolated acts of a genius, nevertheless express a social dimension in mathematics.

It would be dangerous if we attempted to find a straightforward and direct connection between science, on the one hand, and the ideological, cultural, social and economic context, on the other: a "senseless" endeavor, even in the opinion of Friedrich Engels. "Otherwise", Engels wrote to Joseph Bloch, "the application of the theory to any period of history would be easier than the solution of a simple equation of the first degree".[16] Instead, the approach of the present book follows S. R. Mikulinskij. At the 15th International Congress on the History of the Natural Sciences and Technology in August 1977, Mikulinskij said: "The path towards uncovering the mechanisms and laws of the development of science [consists] in understanding the interplay between the objective content of science, the socio-economic, cultural and historical conditions and the personalities involved. Socio-historical practice is decisively influential in this interplay [trans. – H.-J. P.]."[17]

If we keep this basic strategy for uncovering the objective mechanisms of new scientific knowledge in mind, the following view of Graßmann's mathematical work arises: At least seven essential and historically verifiable paradigms of factors had a decisive impact on the development, structure and characteristics of Hermann Graßmann's most important mathematical work, the *Extension Theory* of 1844.

Firstly, one can point to family traditions. Graßmann belonged to an old Pomeranian family of Lutheran ministers. The ties among the family members were very strong. Influenced by Pietism and the Enlightenment, movements which Hermann Graßmann's grandfather had experienced during his days as a student of theology in Halle, the family went through a gradual and somewhat erratic process in which it turned away from religion and towards science. Hermann Graßmann's sons more or less brought this process, which became stronger with every new generation, to an end. All of his sons carried out academic studies in the natural sciences or technology. In the movements of Pietism and the Enlightenment, the German bourgeoisie slowly began to rely on its own practical and intellectual resources. The fact that, in the underdeveloped region of Pomerania, the Graßmann family had preserved and imparted to their children a mindset favoring scientific work was a fundamental prerequisite for Hermann Graßmann's research. It made it possible for him to turn away from theology and towards mathematics in the first place.

Secondly, one must credit what Hermann Graßmann's father, Justus Graßmann, had to offer to his son. This is one of the rare cases in the history of science where the ideas, insights and mindset of the father impregnated the son's scientific vision.

While Justus Graßmann's scientific achievements were hardly remarkable, he possessed a solid philosophical education in the tradition of Leibniz, Kant and the "Naturphilosophie" (philosophy of nature) of German Romanticism.[18] He was a follower of Pestalozzi's pedagogy and of the ideas of a pupil of Pestalozzi, J. Schmid (1809), on how to develop mathematics for elementary-school purposes. The conceptual framework on which Hermann Graßmann's most ingenious achievement, *Extension Theory*, is based, arose from a mixture of Justus Graßmann's perspective on geometry, the Leibnizian combinatorial approach, the Kantian view of mathematics, the dialectical positions of classical bourgeois philosophy in Germany, and Romanticism. It will be one of the main objectives of this book to untangle the complicated web of ideas connecting Hermann Graßmann to his father.

The social and cultural atmosphere in Stettin, the town where Graßmann lived and worked, is a third important factor for the composition and character of *Extension Theory*. The period that immediately followed the War of Liberation of 1813/14 and which ended in the mid 1850s – not very long after the railway connection to Berlin had been completed in 1843 and the bourgeois Revolution of 1848 – was characterized by the flourishing provincial middle class. Romanticism, religiosity and German nationalism were the dominant intellectual tendencies in the town of Stettin. The petit-bourgeois quest for knowledge was underway, and the Stettin Freemason lodge experienced an exceptional rise in membership. Stettin's main secondary school, the "Gymnasium", was the city's scientific and cultural nucleus. It was home to a faculty of professors some of whom managed to do brilliant scientific work despite small-town ignorance and rejection of every external scientific authority that did not follow their own standards. These were teachers with extremely diverse opinions, united only by the Romantic worldview: it was a microclimate particularly favorable to individual creativity. It was the ground on which Hermann Graßmann's confidence in his own capabilities grew, where he became aware that he was capable of completely reorganizing mathematics, starting with the entirely new branch of extension theory. But it was the same petit-bourgeois atmosphere that prompted Hermann Graßmann's brother Robert to write a grotesque 10-volume *Edifice of Knowledge* (1882 – 90). This book claimed to present the totality of human knowledge in a completely novel, putatively "scientific" way. Obviously, this was an endeavor at which Robert Graßmann failed miserably.

Graßmann's *Extension Theory* is an expression of the ambiguity of this particular intellectual milieu. Only a thin line separated provincialism and German nationalism from scientific creativity and brilliance.

The influence of Robert Graßmann, Hermann's brother, is a fourth factor. Hermann Graßmann worked as a teacher in Stettin. Therefore, he developed his ideas far away from universities and centers of scientific research. Robert was his only partner, critic and colleague. The collaboration between the two was so close that today it is often impossible to attribute theoretical concepts to one of the two brothers. For years, every day, they spent many hours collaborating on their scientific projects. The Graßmann brothers debated Schleiermacher's *Dialectic*. They discussed the first and the second version of *Extension Theory* and went over proofs together. For their revision of the foundations of mathematics, they agreed on a division of labor: Hermann would work on extension theory and number theory, Robert on the theory of combinations and logic. The characters of the two brothers differed greatly, and their scientific perspectives were not the same. It comes as no surprise that, for Hermann, who – apart from his correspondence with Möbius – lacked contacts in the scientific community, this collaboration had its brighter and its darker sides. It definitely was one of the reasons why both versions of *Extension Theory* were ignored by the world of mathematics.

Friedrich Schleiermacher's *Dialectic* is a fifth element which, along with Graßmann's father's ideas and approaches, made a decisive impact on the content and structure of *Extension Theory*. Schleiermacher, a philosopher of religion, had been one of the people Graßmann had met at the University of Berlin. As Graßmann put it, he was "infinitely indebted"[19] to Schleiermacher in his scientific outlook. Schleiermacher had introduced Graßmann to the treasure chest of pre-Hegelian dialectics. Drawing his inspiration from Plato, Spinoza, Kant, Schelling, the Romantic philosophy of nature and his own work in the natural sciences, Schleiermacher had showed Graßmann why dialectics was necessary and useful when it came to finding theoretical approaches in mathematics or other disciplines. Schleiermacher also taught his students how these approaches could be transformed systematically into a methodologically coherent theoretical structure.

Schleiermacher's lectures on dialectics (DIAL) were published posthumously two years before Graßmann began his work on *Extension Theory*.[20] Immediately, Hermann Graßmann and his brother fervently began to study them. The work on *Extension Theory* and its systematic construction took place under the immediate influence of Schleiermacher's philosophical studies. A number of factors indicate that Graßmann consciously used Schleiermacher's dialectical method in order to build his mathematical structure and that, in his introduction to the book, he sought to reveal his dialectical approach to the reader. Graßmann's long struggle in finding *Extension Theory's* definite form of presentation, of which he speaks in the introduction, is one of these factors. His insistence that it was inevitable and essential for his work that philosophy be applied to mathematics is another. The brilliance of Graßmann's work is a consequence of the close connection between mathematics and the method of dialectics.

A sixth aspect, namely *Extension Theory's* place in the history of mathematics, follows from the preceding point. Since the 16th and 17th centuries, geometry had lagged behind algebra and analysis in the process of replacing the limited ancient understanding of mathematics. The revolutionary changes in the mathematical conceptualization of geometry had only begun when a foundation for Descartes' analytical geometry was found. In specific ways, geometry became linked to algebraic and analytical methods. For the first time, algebraic and geometrical methods became interwoven in a specific and relatively one-sided way. Reckoning more or less lacked an internal connection to geometry. Given the plurality of possible links between geometry and algebra, this approach led to a dialectical opposition which stimulated the further development of mathematics. On the one hand, prompted by practical needs, this opposition was resolved by the analytical treatment of projective geometry, leading to point, line, and plane coordinates; on the other, it led to the search for a geometrical interpretation of complex and hypercomplex numbers. Driven by problems in mechanics, this search also resulted in vector algebra. Stimulated by the demands of mathematical mechanics, a third solution appeared in attempts at finding and establishing a new, direct connection between algebra and geometry. Leibniz made steps in this direction, and Graßmann's successful foundation of vector and tensor calculi followed this same path. At the same time, he also tackled fundamental questions: he abolished the absolute connection between geometry and metrics, generalized the concept of dimension to *n* dimensions, largely did away with the concept of coordinates and investigated geometrical relationships – all the fundamental problems which, in the 19th century, led to a new level of abstraction in the understanding of geometry, as in F. Klein's Erlangen Program.

Thus Graßmann brilliantly solved the fundamental mathematical problems of his time, following his own approach and profiting from his dialectical mindset.

Finally, we must analyze the individual aspects influencing Graßmann's work and the elaboration of his mathematical ideas. Graßmann's autodidactic learning process is one of those aspects. It protected him from following the beaten paths of science and from being overly influenced by academic trends in mathematics. Another aspect is the fact that Graßmann's interest in mathematics arose at a relatively late point in his life, which gave philosophical thinking time to grow and become influential. Also, the wide range of the secondary-school curriculum which Graßmann was teaching motivated

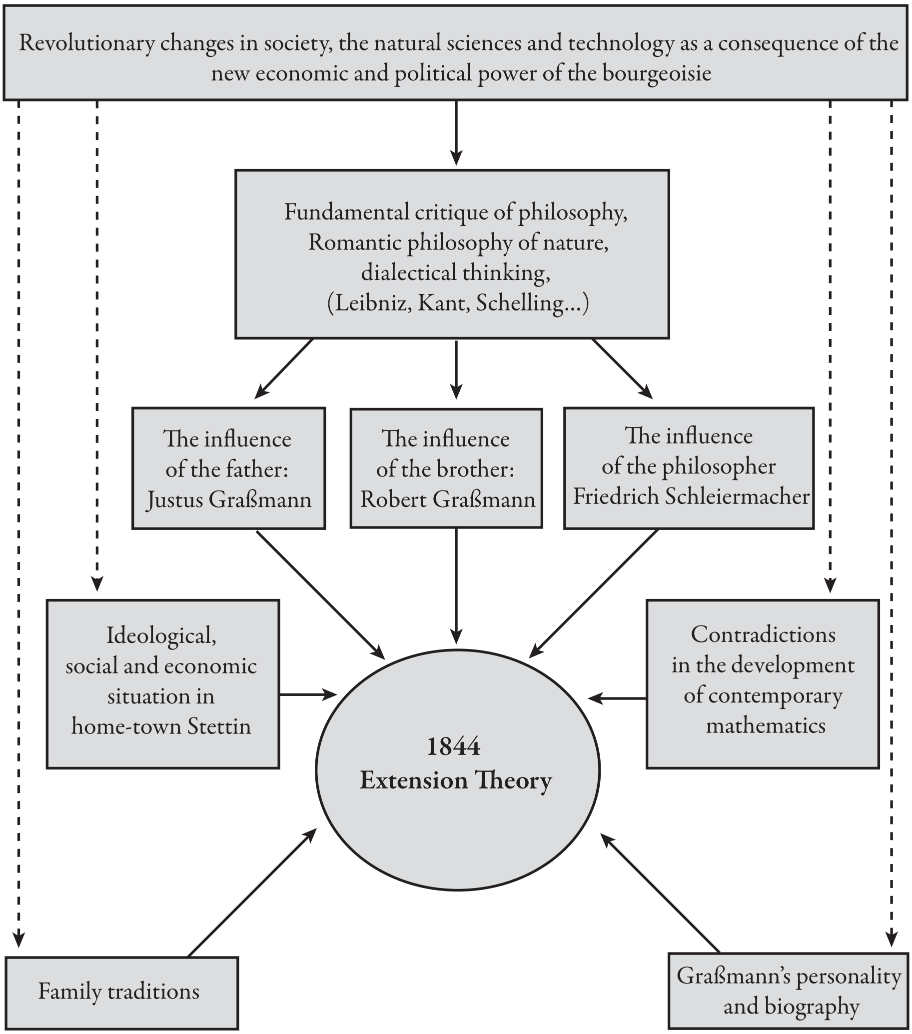


Fig. 1. Influences on Graßmann's *Extension Theory* of 1844

him to penetrate as far as possible the fundamental mechanisms and problems of the sciences. By doing so, he acquired a broad and solid general knowledge. One should not underestimate the importance of solid general knowledge for successful work in specialized scientific fields.

The contours of the socio-economic, cultural, historical and individual mechanisms which determined Hermann Graßmann's work arise from the factors shown above. We see the framework of internal and external conditions which gave Graßmann the maneuvering space he needed to build the *Extension Theory*.

A general insight emerges from this abundance of particular influences: Graßmann's mathematical work was influenced by social history. The fact that he used dialectical thought to solve mathematical problems serves to illustrate this point. In Graßmann's case, the social factors which determined the content of his specific scientific work were, in the first place, intellectual influences going back to Leibniz, Kant, Schelling and Schleiermacher. This is to say that these influences arose from philosophy and dialectics. It is no coincidence that, during this historical period, dialectics appeared in classical bourgeois German philosophy. It was a direct response to the fact that, in Europe, the bourgeoisie was gaining more and more economic and political power. Dialectics was an intellectual reaction to the revolutionary changes that had been taking place in society, the natural sciences and technology since the Renaissance. Graßmann's *Extension Theory* occupies a particular moment in this revolutionary historical process.

Notes

1 Krug 1827, p. 699 – 702.
2 Graßmann used the German letter "ß" in his name. But in scientific literature in German and especially in English, the spelling "Grassmann" is very common.
3 A1, p. 21.
4 A2, p. xiii.
5 Letter from A. F. Möbius to E. F. Apelt, 5 January 1846. Quoted from BIO, p. 101.
6 Letter from A. F. Möbius to C. F. Gauß, 2 February 1847. Quoted from BIO, p. 118.
7 Letter from J. A. Grunert to H. Graßmann, 9 December 1844. Quoted from BIO, p. 103.
8 Letter from E. F. Apelt to A. F. Möbius, 3 September 1845. Quoted from BIO, p. 101.
9 Letter from R. Baltzer to A. F. Möbius, 26 October 1846. Quoted from BIO, p. 102.
10 Cantor 1879, p. 596.
11 GW11-GW33.
12 BIO, p. 314.
13 See Kedrovskij 1974. I would also like to take this opportunity and thank Dr. Steve Russ of the Department of Computer Science of the University of Warwick. I met him

in April 2003 at a workshop on "Knowledge Management and Philosophy" in Lucerne, Switzerland. During a visit to Potsdam he encouraged me to continue my work on Graßmann.

14 H. Graßmann 1996.

15 Schlegel 1878, p. 20.

16 Friedrich Engels, letter to Joseph Bloch, MECW, vol. 49, p. 33.

17 Mikulinskij 1977, p. 104.

18 Schelling's school of "Naturphilosophie" was the dominating force in German philosophical thinking from 1797 to 1830 (see Schelling 1800). The "Naturphilosophie" relied on the notion that natural history and human cognitive processes formed a unity. It based its philosophical views on the concepts of expansion and contraction, space and force. In subsequent sections of the present book, we will use the English term "philosophy of nature" to refer to "Naturphilosophie".

19 Quoted from BIO, p. 21.

20 In the present book, all quotations from Schleiermacher's *Dialectic* refer to the edition published by Jonas in 1839, after Schleiermacher's death (Schleiermacher 1839). This was the edition the Graßmann brothers studied. Schleiermacher had produced different versions of his *Dialectic*. Jonas relied on notes for lectures held by Schleiermacher in 1814 and added material from his lectures of 1818 in the appendix. The edition also contained material from other versions. An English translation of Schleiermacher's *Dialectic* by Terrence N. Tice (Schleiermacher 1996) only uses notes from Schleiermacher's first lecture on Dialectics in 1811. It differs significantly from the Jonas edition studied by the Graßmanns. Therefore, we have not used the Tice translation for the present book.

1 Graßmann's life

1.1 The historical context

The Graßmann family had lived in Pomerania for centuries. Like his father, Hermann Graßmann spent his whole life in Stettin, except for three years in Berlin as a student and one year as a teacher at the Berlin School of Commerce. He was very involved in the life of his hometown. Civic life and culture in Stettin made a lasting impression on Hermann Graßmann's worldview and political positions.

Graßmann's place of birth and home was located in Pomerania, which in the first decades of the 19th century was one of the least-developed German regions. Due to its economic weakness and its strategically important location on the Baltic Sea, Pomerania had repeatedly been affected by dynastic conflicts in the previous centuries.

The Northern Seven Years' War (1563 – 1570), the Thirty Years' War (1618 – 1648), the Polish-Swedish War (1655 – 1660), the Great Northern War (1700 – 1721) and other conflicts devastated the country, brought terrible poverty and suffering to the working population and set the cities back in their economic development.

In 1677, West Pomerania, which up to that point had been ruled by the Swedes, fell into the hands of the elector of Brandenburg. The city of Stettin had been besieged for five months (23 July – 27 December 1677), badly destroyed and conquered by the elector's troops. After the peace treaty of Saint Germain (1678), France used its influence and Stettin was returned to the Swedes. For a long time, the seaport town of Stettin struggled with the damage the war had caused.

Following the Great Northern War (1700 – 1721), Frederick William I (1713 – 1740) managed to realize the old political goal of the state of Brandenburg and gained "access to the sea", putting it "in a position to participate in worldwide commerce".[1] But despite a large number of improvements and extensions, the Stettin seaport never lived up to expectations. Under Frederick William I and Frederick the Great (1740 – 1786), the administration of Pomerania was adapted to the needs of militant feudal absolutism. The city lost whatever had remained of its independence. From 1724 to 1740, the city became one of the largest fortified settlements in the country (43 million stones were used to build the ramparts!).[2]

In conformity with the militant feudal absolutism of the kings of Prussia, manufactories were submitted to state regulations and supported by franchises, privileges and monopolies. Industry was reorganized to meet the needs of the military and the royal court. This made the lower and middle classes economically dependent on the aristocracy and the military. Sea trade created tight economic ties between merchants and local landlords, but also between merchants, the royal court and the military. Almost all agricultural resources were destined for export. Imports consisted mostly of sophisticated manufactured goods.

The combination of absolutistic centralism and economic underdevelopment led to uneven and badly coordinated development in Stettin. As a consequence, many of the newly founded industries and business ventures were short-lived. Business could not develop freely under the bureaucratic guidelines it was confronted with, the middle class remained chained to the feudal system, and productive potential was lost.

Fig. 2. Stettin in the process of becoming one of Prussia's largest fortified cities

Instead of awakening "individual resources and independence", instead of "free citizens", the reforms and regulations of absolutism created "subjects".[3] Protectionism led to small-minded, petit-bourgeois conservatism. F. Engels condemned the "inert indifference" and "limp, if not servile, torpor"[4] he had encountered in northern Germany.

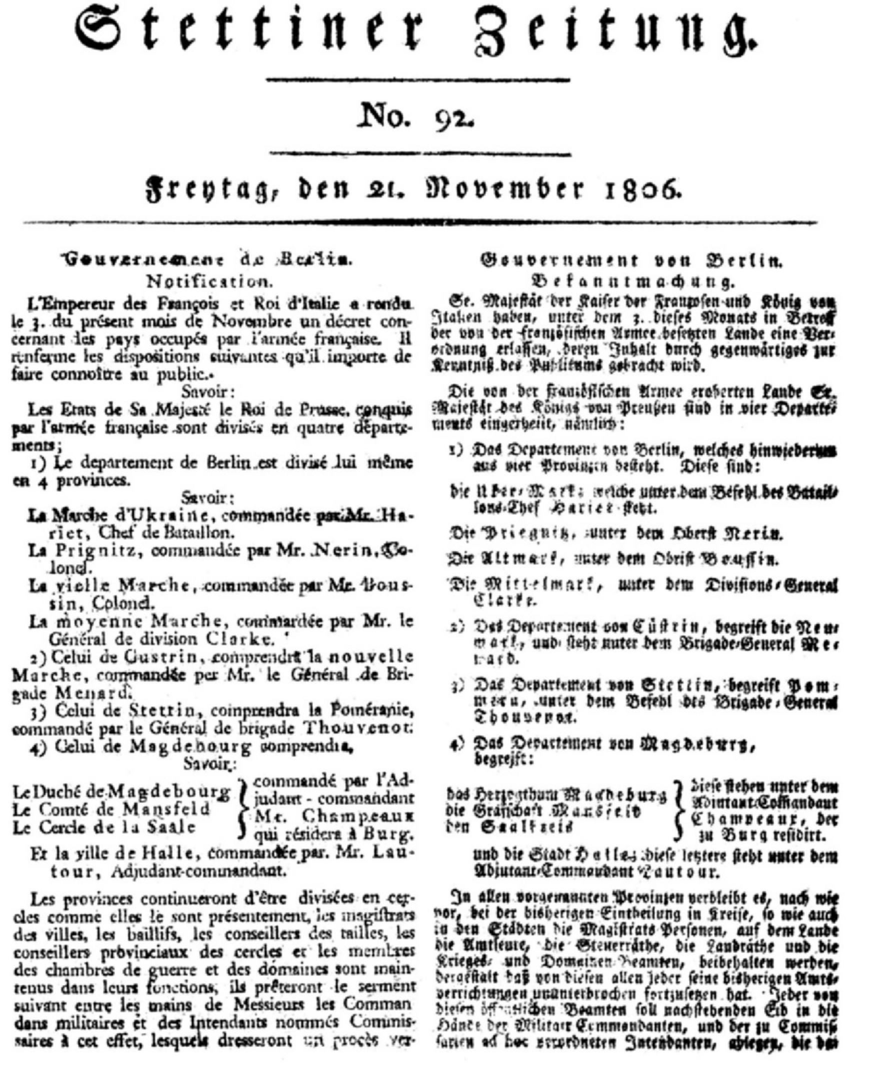
Apart from its importance as a seaport and military base, Stettin was also the seat of the Pomeranian political administration. Most government employees had attended the city's academic secondary school. This school, the "Gymnasium", was the most important of three secondary schools in the Prussian part of Pomerania at the time of Frederick the Great. It had been founded in 1662.[5] Despite attempting a display of academic glamour, such as an alternating presidency and elective lectures, the "Gymnasium" of Stettin was mediocre. Things improved in 1805 when the two schools offering a higher education in Stettin, the "Rats-Lyceum" and the "Akademisches Gymnasium", merged to form the city's "United Royal and Municipal Gymnasium". By the mid-19th century, the school was in an upswing.[6]

But even then, the social stratification of the city was patriarchal and characterized by strict dividing lines between the aristocracy and the bourgeoisie, between civilians and the military, government and citizenry.[7]

Starting in 1806, Napoleonic troops occupied the fortified city for seven years. This was a rupture in its fragile economic development. The occupying army exerted considerable economic pressure on the city, draining it of its resources for the benefit of the French upper classes and ruining it slowly. At the time, total losses were estimated at about 5 ½ million taler.[8]

Stettin experienced a renaissance thanks to the new nationalist movement and the bourgeois "half-and-half reformers"[9] who managed to loosen the grip of absolutism on the administration and economy of the city. The petite bourgeoisie and the middle class only slowly became interested in the city's public affairs, while the government officials served as middle-men between the aristocracy and the bourgeoisie. But the academic intelligentsia surrounding the "Gymnasium" (Bartholdy, Sell, Koch, J. Graßmann) became politically active.

Between 1816 and 1831, August Sack was the governor of Pomerania. He was a former protégé and close confidant of the important Prussian politician Karl vom Stein and actively engaged in freeing the country from the Napoleonic chokehold. Sack had begun his career trying to defend and continue reforms in the Rhine Province. In 1814, he became the general governor of the Lower Rhine Province and called for a fundamental reform of the school system, calling "primary schools the 'essential source of all national education and national will'" and asking "'every educator, every scholar, every philanthropist who cares about this sacred duty' to collaborate."[10] But the vigorous monarchist rollback increasingly limited his range of activity. In 1816, Sack was sent to Pomerania, but he remained true to his beliefs, even though his effectiveness was greatly



Stettiner Zeitung.

No. 92.

Freytag, den 21. November 1806.

Gouvernement de Berlin.
Notification.

L'Empereur des François et Roi d'Italie a rendu le 3. du présent mois de Novembre un décret concernant les pays occupés par l'armée française. Il renferme les dispositions suivantes qu'il importe de faire connoître au public.

Savoir:

Les Etats de Sa Majesté le Roi de Prusse, conquis par l'armée française sont divisés en quatre départements;

1) Le departement de Berlin est divisé lui même en 4 provinces.

Savoir:

La Marche d'Ukraine, commandée par Mr. Hariet, Chef de Bataillon.

La Prignitz, commandée par Mr. Nerin, Colonel.

La vielle Marche, commandée par Mr. Boussin, Colonel.

La moyenne Marche, commandée par Mr. le Général de division Clarke.

2) Celui de Custrin, comprendra la nouvelle Marche, commandée per Mr. le Général de Brigade Menard.

3) Celui de Stettin, comprendra la Poméranie, commandé par le Général de brigade Thouvenot.

4) Celui de Magdebourg comprendra,

Savoir:

Le Duché de Magdebourg, Le Comté de Mansfeld, Le Cercle de la Saale — commandé par l'Adjudant-commandant Mr. Champeaux qui résidera à Burg.

Et la ville de Halle, commandée par Mr. Lautour, Adjudant-commandant.

Les provinces continueront d'être divisées en cercles comme elles le sont présentement, les magistrats des villes, les baillifs, les conseillers des tailles, les conseillers provinciaux des cercles et les membres des chambres de guerre et des domaines sont maintenus dans leurs fonctions, ils prêteront le serment suivant entre les mains de Messieurs les Commandans militaires et des Intendants nommés Commissaires à cet effet, lesquels dresseront un procès ver-

Gouvernement von Berlin.
Bekanntmachung.

Se. Majestät der Kaiser der Franzosen und König von Italien haben, unter dem 3. dieses Monats in Betreff der von der französischen Armee besetzten Lande eine Verordnung erlassen, deren Inhalt durch gegenwärtiges zur Kenntniß des Publikums gebracht wird.

Die von der französischen Armee eroberten Lande Sr. Majestät des Königs von Preußen sind in vier Departements eingetheilt, nämlich:

1) Das Departement von Berlin, welches hinwiederum aus vier Provinzen besteht. Diese sind:

die Ukermark, welche unter dem Befehl des Bataillons-Chef Hariet steht.

Die Priegnitz, unter dem Oberst Nerin.

Die Altmark, unter dem Obrist Boussin.

Die Mittelmark, unter dem Divisions-General Clarke.

2) Das Departement von Cüstrin, begreift die Neumark, und steht unter dem Brigade-General Menard.

3) Das Departement von Stettin, begreift Pommern, unter dem Befehl des Brigade-General Thouvenot.

4) Das Departement von Magdeburg, begreift:

das Herzogthum Magdeburg, die Grafschaft Mansfeld, den Saalkreis — diese stehen unter dem Adjutant-Commandant Champeaux, der zu Burg residirt.

und die Stadt Halle; diese letztere steht unter dem Adjutant-Commandant Lautour.

In allen vorgenannten Provinzen verbleibt es, nach wie vor, bei der bisherigen Eintheilung in Kreise, so wie auch in den Städten die Magistrats-Personen, auf dem Lande die Amtleute, die Steuerräthe, die Landräthe und die Krieges- und Domainen-Beamten, beibehalten werden, dergestalt daß von diesen allen jeder seine bisherigen Amtsverrichtungen ununterbrochen fortzusetzen hat. Jeder von diesen öffentlichen Beamten soll nachstehenden Eid in die Hände der Militair-Commendanten, und der zu Commissarien ad hoc verordneten Intendanten, ablegen, die den

Fig. 3. Title page of the local Stettin newspaper during the Napoleonic occupation of the city

reduced. His administration brought gradual economic improvement to the country. On his initiative, merchant guilds and local trade associations were abolished, and a Chamber of Commerce was created in 1821.[11] Even though membership was obligatory, thereby violating legislation on the freedom of trade, Sack's innovation had a lasting impact on the development of Stettin.[12] After Sack's death on 28 June 1831, the merchants of Stettin erected a monument in his honor in 1833.[13]

Even though two new piers were built between 1817 and 1829 and the rivers Oder and Swine were dredged, Stettin, the largest Prussian port, had hardly any importance for the rest of the country. It could not compete with the ports of Bremen, Lübeck and Hamburg. Only after 1839 did wholesale rates in Stettin increase significantly.[14] Until the mid 19th century, internal trade was more important than export.

In 1823, the founding of the "Knightly Private Bank of Pomerania" was yet another manifestation of the tight economic ties between landlords and merchants. An association of conservative landlords and wealthy merchants, the bank was the dominant financial force in Stettin until 1846 and the first German bank to receive permission from the Prussian government to issue its own bank notes.[15]

But economic development was sluggish. For a long time, the "Provincial Sugar Factory of Pomerania", founded in 1817, remained the epitome of modern industrialism and the main factory and stock company in Stettin.

The Pomeranian government paid special attention to the shipyards. The first Prussian gunboat was built in Stettin by the conservative head of the industry, naval engineer and shipyard-owner A. E. Nüscke (1817 – 1891). This was a first contribution to what after 1848 would become the Prussian navy.[16]

In the years after 1825, the Stettin wool market became economically relevant in northern Germany. At the time, 430 producers supplied it with 500 tons of wool.[17]

Stettin's economy was increasingly confronted with competitors when roads were constructed between 1822 and 1829 and the railroad connection to Berlin was completed in 1843. Businesses based in Berlin became its main competitors. In the 1850s, the major branches of industry were still in an early phase of development. In contrast to other parts of Germany, capitalist production had not yet marginalized the crafts in the region, even in the late 19th century. Capitalist enterprises such as shipbuilders, iron and brick companies established themselves alongside the river Oder.

The Hohenzollerns shielded Stettin from outside influences, and the city's development was dominated by petit-bourgeois provincialism. Engels might very well have meant Stettin when he wrote in 1847: "The petty bourgeoisie represents inland and coastal trade, handicrafts, manufacture based on handwork – branches of industry which operate within a limited area, require little capital, have a slow turnover and give rise to only local and sluggish competition."[18]

The petit-bourgeois attitude towards politics also reflected the economical outlook of this social class: "Next to the peasants, it is the most pathetic class that has ever meddled with history. With its petty local interests, it advanced no further even in its heyday (the later Middle Ages) than to local organizations, local struggles ..."[19] Shortly before the bourgeois Revolution came to Germany, the citizens of Stettin supported the conservatives in the 1848 elections for the Prussian National Assembly.[20]

Stettin's academic intelligentsia, most of which was connected to the secondary school, mirrored the petit-bourgeois atmosphere and the underdeveloped economic situation of the city. In the first half of the 19th century, the Stettin "Gymnasium" was the city's main ideological, scientific and cultural institution. It had taken until the late 18th century for Enlightenment thought to establish itself in Stettin, bringing with it Freemasonry and encyclopedic projects. The Napoleonic occupation drove the city's educated elite towards nationalism. The elite resented all French influences and delved into German history. Given their deep monarchic sympathies, especially among the school's employees, this attitude could only manifest itself as conservatism or – in special cases – as liberalism. This range of individual attitudes found common ground in Romanticism, which in the city's petit-bourgeois, patriarchal atmosphere was synonymous with reactionary political tendencies and scientific parochialism.

Romanticism was the first bourgeois reaction to a feeling of crisis. It expressed a radical, but seemingly futile criticism of traditional feudal structures and of the new middle class, turning to Antiquity and the Middle Ages for inspiration. Therefore, it showed both progressive and reactionary tendencies.

Romanticism renewed national pride by evoking the Middle Ages and created enthusiasm for heroic acts of liberation by rediscovering popular poetry and songs (the brothers Grimm, C. Brentano, E. M. Arndt, Th. Körner, H. v. Kleist, etc.), but it also glorified times when "landlords and priests"[21] represented the dominating social class. The

Carl Loewe (1796 – 1869), the ballad composer, is considered one of the last great masters of Neo-Romanticism in music.[22]

He was born on 30 November 1796 in Löbejün (near Halle) as the twelfth child of cantor Andreas Loewe. He was a choirboy in Köthen and attended secondary school in Halle. In 1817, he became a student of theology. His encounter with Weber in Dresden was decisive for his further development as a musician. In 1820, he came to Stettin as the city's music director. There, he was initially accommodated in living quarters provided by the city, namely in Justus Graßmann's apartment in the old "Gymnasium". In the evening hours, the Graßmanns invited Loewe to participate in family life.[23]

He became an intimate friend of Justus Graßmann and Professor Giesebrecht (all three were members of the Stettin Freemasons, where Carl Loewe gave many public concerts). He carried out scientific work on astronomy and acoustics with Justus Graßmann. He taught Hermann Graßmann to play the piano and bass.[24]

This is how Carl Loewe described his feelings about Justus Graßmann: "On the evening of May 29th, there was a gathering at Professor Graßmann's, a highly educated and sophisticated man. Ever since my arrival, we have been the closest of friends. The many interests we have in common always give us things to discuss. – Many of the city's educated men, but also the ladies, listen to his lectures on physics. I learned new things about the theory of sound..."[25]

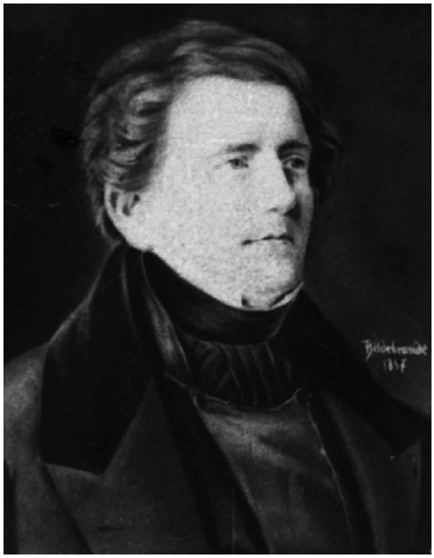


Fig. 4. Carl Loewe (1796 – 1869)

monarchist reactionaries were interested in this Romantic tendency and transformed it into a Romantic theory of the state, a project especially dear to Fredrick William IV.

Religion was back at the center of attention. Connected to the philanthropic movement, which created a wide range of charitable organizations, religion regained lost ground. The Stettin Freemasons, whose number had increased from 35 in 1798 to 117 in 1818,[26] brought together all of the city's major scientific and cultural personalities and were part of these tendencies. Apart from liberal-minded Justus Graßmann, H. Hering and C. G. Scheibert, two conservative followers of the house of Hohenzollern, became lodge Presidents.

When the composer Carl Loewe moved to Stettin in 1820, music also came under the sway of Romanticism. Art and history clubs flourished.[27]

For a long time, the "Gymnasium" remained true to its academic ideals – offering, for example, lectures for doctors, lawyers and philologists[28]. In its small-town isolation, it experienced a phase of scientific excellence, which nevertheless never went beyond the city walls. Apart from Romanticism, the faculty members possessed no common religious or philosophical background. Philosophical affiliations coexisted side by side – whether to Plato or Aristotle, Kant, Fichte, Hegel, Spinoza, or Schleiermacher.[29] During its most productive phase, the Stettin "Gymnasium" was attended by Hermann Graßmann, the physicist Rudolf Clausius and the poet Robert Prutz. But it had also been home to conservative politicians such as the ministers v. Hertzberg, v. Raumer and the Count of Schwerin, who actively resisted the 1848/49 revolution. Noteworthy conservatives such as the presidents of the Prussian House of Representatives and of the German Reichstag v. Levetzow and v. Köller also attended the school in their youth.[30]

When travel connections to Berlin improved and capitalism boomed in the years following the 1848 revolution, the "Gymnasium" quickly lost its academic importance and unique profile.

Graßmann was extraordinarily attached to his hometown. Given the economic and ideological conditions mentioned above, this attachment is the key to understanding his worldview and political convictions. While other progressive thinkers from Pomerania such as the revolutionary democratic poet and literary historian Robert Eduard Prutz (1816 – 1872), the physicist Rudolf Clausius (1822 – 1888) and the medical scientist Rudolf Ludwig Virchow (1821 – 1902) eventually turned their backs on the region's unpromising situation, Graßmann never left. His great achievements and his limitations must be judged with this in mind.

H. Müller on the peculiarities of Stettin's scientific upswing in the mid 19th century

"Stettin, even though it was the most important city for German Baltic Sea trade, a seat of local political administration and a fortified military city, was not a place where the scientific spirit was likely to develop. ... Much went unheard in the sleepy bourgeois circles, and many ideas attacked the thick city walls in vain or never surmounted the ramparts. ...

Making up for this, a benign spirit blessed certain men with a special gift. Some of these men, such as Robert Prutz, became known and respected in Germany. But most of them were ignored by the masses, even though the idealism, righteousness and intellectual depth of their works place them among the ranks of our finest. Even today we are uplifted by Ludwig Giesebrecht's poetry, ... by Karl Löwe's ballads, ... by Ferd. Oelschläger's quartets. ... Kugler, the poet, the historian Schmidt, Martin Plüggemann, whom Wagner called Löwe's most important student, the original pedagogue Calo, and Hermann Graßmann, equally erudite in mathematics and the science of Sanskrit, are also among the intellectual giants of Stettin's classical period.

They all met in the brotherhood of the 'Lodge of the Three Circles', and this is where they developed their original thinking. But since they never paid attention to other currents of thought and never participated in life outside the city, they also remained confined to this small space. They had little or no contact with similar endeavors in other parts of Germany and therefore have not received the attention they deserve."[31]

1.2 The family: traditions and relatives

Hermann Günther Graßmann (1809 – 1877) was from an old family of Protestant ministers. This background shaped Graßmann's entire life.[32]

In the mid-18th century, interest in scientific work arose in this family, which had been firmly rooted in underdeveloped Pomerania for centuries. Gottfried Ludolf Graßmann (1738 – 1798), a theologian and Hermann Graßmann's grandfather, was extremely open to scientific problems. His attitude towards scientific research was linked to the movement of Pietism and to German Enlightenment philosophy.

Gottfried Ludolf Graßmann made important contributions to agricultural science in his day and was the originator of a development which led the family away from theology and towards science. With the subsequent generations of the Graßmann family, this movement, which was full of contradictory tendencies, made its way into the future.

Hermann Graßmann's grandfather was born on 3 April 1738 in the town of Landsberg, on the river Warthe.[33] He lost his father in early childhood, and from that moment on he was obliged to help his widowed mother with the farm chores. He was 22 years old when, after having earned the necessary money as a choirboy, he began his studies of theology at the University of Halle. Since its foundation in 1694, the University of Halle had been the most important stronghold of the German Enlightenment and of Pietism. This is why Gottfried Ludolf Graßmann's three years in Halle

Fig. 5. Gottfried Ludolf Graßmann (1733 – 1798), Hermann Graßmann's grandfather

are so important. In Halle, August Hermann Francke, Christian Wolff and Sigmund Jakob Baumgarten, among others, had defended an enlightened and free understanding of religion and the world against Lutheran orthodoxy.

The German bourgeoisie favored the religious views of Pietism because it no longer strove exclusively for heavenly salvation but also – and mainly – pursued worldly goals. In Pietism, G. L. Graßmann found a worldview which provided the basis for his later work on the theory and practice of agriculture and theology.

After finishing his studies, he returned to Pomerania, where Lutheran orthodoxy was still strong, and became a village pastor. He developed strong friendly ties to his rural congregation and decided never to leave his parish, despite various offers to do so.

In 1876, Robert Graßmann gave the following description of his grandfather: "Everybody who knew him liked him for his friendly and gentle way with people, and because he had been born with a warm heart. He truly loved religion and was content in serving it, and when he spoke of religion a benign flame filled him with life; people in the entire

Eine erlauchte und freye ökonomische Gesellschaft haben dem Publikum folgende Preisfrage, zur Beantwortung vorgeleget:

„Nach den verschiedenen Eigenschaften der „Felder, in den Gouvernements Ingermann„land, Moskow, Belgorod, und denen Pro„vinzen Pleskow, Orel und Smolensk, wel-

Graßm. Best. d. Land. rc. A che

2

„che in die, bey der Landwirtschaft, vorge„schriebenen vier Nummern eingetheilet wer„den, genau zu bestimmen, wie viel Land „eigentlich zum Hause, zum Akker und zu den „übrigen Bedürfnissen, einer aus zween Ar„beitern und zwoen Weibern und der verhält„nismässigen Anzahl von allen abgelebten Leuten „und Kindern bestehenden Bauerfamilie, er„fordert werde, damit diese Familie ihren „reichlichen Unterhalt haben und auch alle und „jede Gefälle richtig abtragen könne?

Das Gemeinnützige, so in dieser Preisfrage lieget, mus einen jeden der sich einiger Kentnis in diesem Fache bewust ist und dabey Empfindungen, die zum Wohl der Menschen abzielen, besitzet, billig ermuntern seine hierin erlangten Einsichten zu deren Auflösung anzuwenden. Kaum können seine erlangten Kentnisse besser als hier genutzet werden: denn ich kenne noch keine ökonomische Gesellschaft, ob mir gleich die Einrichtung der mehresten bekant, welche so einmüthig auf das Wohl ihrer Mitbürger bedacht und zugleich eben so mächtig ist, das nützliche der Vorschläge in so weitläuftigen Gouvernements und Provinzen, ohne Verzug in Ausführung zu bringen.

Ich

Fig. 6. A text Gottfried Ludolf Graßmann submitted in an essay contest in 1776

Justus Günther Graßmann,
* 19. 6. 1779 in Sinzlow,
† 9. 5. 1852 in Stettin,
Professor in Stettin.

Johanna Friederike Luise Medenwaldt,
Frau des Professors Justus Günther Graßmann,
* 30. 10. 1785 in Klebow,
∞ 29. 5. 1804,
† 2. 6. 1841 in Stettin.

Fig. 7./8. Hermann Graßmann's parents

region appreciated this, and they liked him even more for his simple and accessible sermons. … When he was offered more land, a proposal much to his advantage, he rejected it simply because he knew that this could provoke hard feelings between his parish and himself, something he was determined to avoid."[34]

Apart from his pastoral work, he soon began to engage in extremely successful agricultural experiments. He became the editor of a Berlin-based journal on agricultural science, which appeared from 1792 to 1794.[35] He won a prize from the Berlin Royal Academy of Sciences for his *Treatise on the General Aspects of Feeding Animals in the Stable, and on Whether to Maintain or Abolish Fallow* (Berlin 1788). In 1779, he won the Russian Imperial Academy of Science's prize for a tract called *Which Good and Inexpensive Measures can be taken to improve the Durability of Ship Wood* (St. Petersburg/Leipzig 1784).[36]

His achievements were rewarded when he was named royal commissioner of cultural affairs in Pomerania. Gottfried Ludolf Graßmann became a loyal servant of Prussian absolutism and, a sincere believer in the monarchy, advocated reforms in the feudalistic structures of agriculture, which were being discussed at the time. The conviction that both bourgeois emancipation and a close alliance with the monarchy were possible was widespread in German petit-bourgeois circles until the 1840s. This conviction would also be very important for future members of the Graßmann family.

Justus Günther Graßmann, Hermann Graßmann's father, was born on 19 July 1779. Justus Graßmann's scientific interests shaped his entire life. Also attending the University of Halle to study theology, like his father, he admitted that: "... I have always been attracted to the exact sciences, and therefore I diligently studied mathematics with Klügel and Gilbert, and from the latter I also learned about physics."[37]

During his student days in Halle (1799 – 1801), the philosophy of nature was at its peak.[38] The important university at Jena was a strong influence on the intellectual climate in Halle. Achim von Arnim, the Romantic poet, physicist and philosopher of nature, was working and publishing in Halle when Justus Graßmann began his studies.[39] Von Arnim was especially interested in the theory of combinations and crystallonomy, the fields Justus Graßmann was most attracted to during the rest of his life.

After finishing his studies, following in his father's footsteps, Justus Graßmann became a private teacher. In March 1802 he passed his theological exams and was sent to the town of Pyritz as a secondary-school teacher. He also preached in the local church. In 1806, he gave up preaching and took up a position in the recently established "United Royal and Municipal Gymnasium" in Stettin (as the so-called second teacher of mathematics), where he stayed for the rest of his life.

Justus Graßmann's first days at the Stettin "Gymnasium" coincided with the fortified city's shameful capitulation to an advance guard of only 800 French horsemen[40] on 30 October 1806 and the subsequent occupation of the city by Napoleonic troops. National pride was on the rise, and Justus Graßmann propagated progressive bourgeois worldviews in the shape of Christian morality and brotherliness. He became actively engaged in city affairs. Apart from his duties as a teacher, he participated in various administrative capacities, working to improve the elementary-school system and the lot of the poor. Imbued with the spirit of Schleiermacher and Humboldt, who believed that the renewal of the educational system would contribute to a German national renaissance[41], Justus Graßmann began working at the Stettin seminar for teachers in August 1812. This seminar had been reestablished by the local school councilor Bartholdy, a friend of Schleiermacher.[42]

In 1813, Justus Graßmann – father of four children – volunteered for the Prussian army. His wife, Johanna Friederike Medenwald, whom he had married in 1804, fled the city, which was occupied by the French and besieged by the Prussians, seeking refuge in her mother's house. Among their children was Hermann Graßmann, born on 15 April 1809.

Discharged from military duty, Justus Graßmann returned to work in May 1814 and took a position as the school's leading teacher of mathematics after the death of school councilor Bartholdy (26 May 1815). Shortly thereafter, he received the title of professor.

Justus Graßmann was also affected by petit-bourgeois Romanticism and religious enthusiasm which blossomed after liberation from Napoleonic rule. He joined the Stettin Freemasons and quickly became one of their most active members. He also got involved in a wide variety of patriotic clubs. Carried away by nationalistic and religious enthusiasm – like many other "good-natured enthusiasts", "Germanomaniacs by extraction and free-thinkers by reflexion"[43] as Marx put it – Justus Graßmann fervently supported bourgeois humanism and the idea of local progress. This attitude also manifested itself, apart from the activities mentioned above, in his involvement with the Stettin Philhellenistic Society. He became one of the leading members of this group, which wanted to provide humanitarian support to Greek freedom fighters.

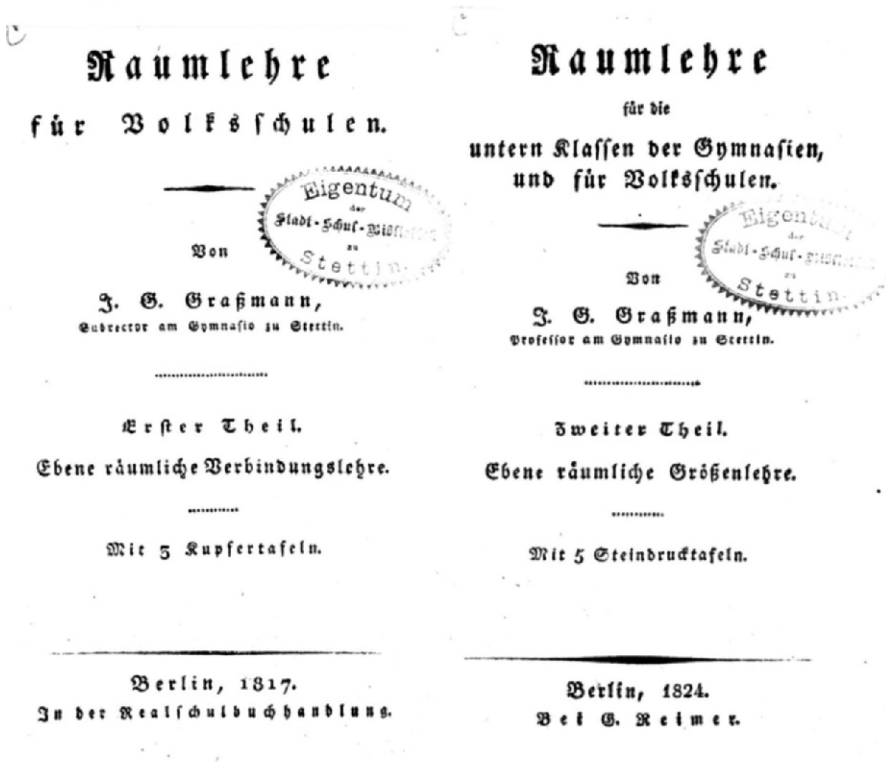
When the "Turnverein" gymnastics movement[44] came to Stettin in 1818, "only 16 participants were on the list. The teachers refused to take part, with one exception. … Professor Graßmann was the only one to get involved."[45] In January 1819, J. Graßmann became a member of the Stettin "Turnverein".[46]

But despite this wide range of practical activities, he still found time for scientific work. Incorporating Pestalozzi's humanistic views, he wrote three mathematical textbooks for elementary and secondary schools[47] which were praised by the important pedagogical theorist Diesterweg.[48]

Justus Graßmann and the schools for the poor in Stettin

"… but we will have to mention that his studies and his contributions to the further education of teachers (together with his brother, school councilor [Friedrich Heinrich Gotthilf – H.-J. P.] Graßmann, and councilor Bartholdi) helped bring the fruitful elements of Pestalozzi's pedagogy to the schools for the poor, for which he was responsible, and therefore also to regular schools. Also, his collaboration with the city's administration contributed to improving the city's school for the poor significantly, provoking bashful envy on the part of the citizens. This prompted the citizens to make great concessions to the city's schools and contributed to the current excellence of the municipal schools."[49] (C. G. Scheibert 1853)

In 1835, J. Graßmann also founded the Physics Society, where he held lectures and carried out experiments. He published a number of suggestions on how to improve scientific instruments. J. Graßmann and his friend the Romantic composer C. Loewe became involved in astronomical observations. Besides all this, he published a quite philosophical treatise on the theory of number (ZL) and wrote a book on crystallography (KRY). Its system of indexes later became influential in the scientific world.[50]


Raumlehre
für Volksschulen.
Von
J. G. Graßmann,
Subrector am Gymnasio zu Stettin.
Erster Theil.
Ebene räumliche Verbindungslehre.
Mit 5 Kupfertafeln.
Berlin, 1817.
In der Realschulbuchhandlung.

Raumlehre
für die
untern Klassen der Gymnasien,
und für Volksschulen.
Von
J. G. Graßmann,
Professor am Gymnasio zu Stettin.
Zweiter Theil.
Ebene räumliche Größenlehre.
Mit 5 Steindrucktafeln.
Berlin, 1824.
Bei G. Reimer.

Fig. 9./10. The title pages of Justus Graßmann's *Geometry*

The scientific and philosophical approaches in these works would become the decisive point of departure for his son Hermann's scientific inquiries.

J. Graßmann's son-in-law, C. G. Scheibert, described his father-in-law thus: "The bright light of the Gospel, which shone on everything and everyone he laid his eyes on, gave him a clear picture of all things, great and small. It heightened his boundless love for the fatherland and the monarchy and brought a comforting and enlivening warmth to his relations with others, whether they were close to him or not. … He was an utterly harmonious person in his ethics and his scholarship. He studied a great deal, but he only accepted as his personal knowledge what he had completely understood, had worked on with the entire range of his thinking and woven into his scientific worldview. Thus, whatever he learned from others could enter into his own original ideas."[51]

This epistemological particularity reappeared in Hermann Graßmann, his son.

Justus Graßmann's development was strongly influenced by the movement of liberation, which aimed to free Germany from Napoleonic rule. His education attracted him to bourgeois humanism and he was strongly attached to the natural sciences and mathematics. The events which occurred between 1806 and 1813 changed his worldview, giving it a nationalist, religious and Romantic direction.

In the midst of economic problems, J. Graßmann fought a liberal and patriotic battle for local progress and was firmly convinced of the "positive moral values" of the monarchy.

1.3 Graßmann's youth and university years

Hermann Günther Graßmann was born in Stettin on 15 April 1809, Euler's birthday, as the third child of Justus and Johanna Graßmann.

Justus Graßmann and his wife Johanna Friederike (*née* Medenwald) had twelve children:[52]

1. <u>Luise</u> Mathilde (1805 – 1807);
2. Karl <u>Gustav</u> (1807 – 1841), minister;
3. <u>Hermann</u> Günther (1809 – 1877), professor at the Stettin "Gymnasium";
4. <u>Alwine</u> Marie (1810 – 1834), married to Stettin headmaster Christian Heß;
5. <u>Adelheid</u> (1812 – 1861), married to headmaster and regional school director Carl Scheibert;
6. Siegfried <u>Robert</u> Ludolf (1815 – 1901), teacher, printer and newspaper editor;
7. Emma Friederike <u>Therese</u> (1817 – 1867), married to justice official Ferdinand Alexander Wegeli;
8. <u>Justus</u> Gotthold Oswald (1818 – 1893), churchwarden;
9. <u>Karl</u> Friedrich Eduard (1820 – 1847), student of theology;
10. <u>Heinrich</u> August Friedrich (1824 – 1855), teacher;
11. <u>Johanna</u> Elise (1827 – 1861), teacher;
12. <u>Sophie</u> (1830 – 1834).

His youth was overshadowed by the Napoleonic occupation of Stettin. Hermann Graßmann's first teacher was his mother, since his father had volunteered for the Prussian army in 1813 and the family had sought refuge in a village nearby. After the family had been reunited in the summer of 1814, the boy attended a private school and then, after sixth grade, spent 7 ½ years at the "United Royal and Municipal Gymnasium", where his father was a professor.

When Hamilton, whose later mathematical and scholarly work was so closely related to Graßmann's, was a young boy, he had the reputation of being a prodigy: "At three years of age he read English quite well and was good at arithmetic; when he was four, he was a good geographer; at five he read and translated Latin, Greek and Hebrew ...; at eight he also mastered Italian and French and was able to improvise fluently on the beauty of the Irish landscape in Latin hexameters ...; finally, before he was ten, he began to prepare what would later be his extraordinary knowledge of Oriental languages by making first steps in Arabic and Sanskrit. ... When he was thirteen, Hamilton could boast that he had learned one language for each year of his age ..."[53]

Hermann Graßmann was a different case altogether. He was anything but a prodigy. His fragile disposition forced him to fight for self-esteem and intellectual force. The boy's talent faced three obstacles: timidity, forgetfulness and dreaminess.

We can learn much about Graßmann's development from two curricula vitae[54] which he wrote on the occasion of university exams. In these two texts he was extraordinarily objective and quite tough on himself. He insisted that he had showed little talent during his first years at school, had been incapable of intellectual effort and that his power of reasoning and recollection had lagged behind. "[Graßmann's] father often said that he would bear it stoically should his son become a gardener or a craftsman as long as his son was good at his work and labored honorably to help his fellow man."[55]

Looking back, Graßmann called the years before his fourteenth birthday a time of "sleepiness" and "daydreams ... provoked by his intellectual sluggishness ..."[56] These daydreams were visions of prosperity, of being loved and admired, full of vanity and self-aggrandizement. They had made him "deaf to the outside world". When he played with older boys, he either refrained from participating actively or did whatever he was told, without ever acting on his own. Hermann Graßmann usually spent his vacations in the countryside with relatives who, for the most part, were ministers – and he dreamed of a "pleasant life with no worries".[57]

But his life changed radically when puberty arrived. As he remarked himself, the phase of "awakening" from his "state of slumber" set in when he was 14 years old. Open to ethical questions, he began to think about his way of life. His confirmation teacher encouraged young Graßmann to awaken his powers of intellect. "At the time, I was in eighth grade", he reports in his curriculum vitae, "and we were supposed to memorize a poem. I had chosen a section from Klopstock's *Messias*; having recently decided to awaken my mental powers, I applied them to this task. I decided to commit myself completely and be fully aware of what I was trying to memorize, and I was surprisingly successful in doing so ... It was only now that I realized that it had been my own fault when, earlier on, I hadn't been able to grasp things well, and it was only now that I became aware that I had to, and *could*, wake up. Filled with self-confidence ... I then decided to awaken my entire life. And it is a fact that, thereafter, the teachers praised my essays in German class, which so far had been justly criticized; it seems as if only then did I really begin to think. I also became livelier in everyday life."[58]

Religiosity was important in the Graßmann family and, in those years, it was on the rise everywhere. Understandably, Hermann Graßmann's thoughts on morality and ethics, as well as his general worldview, were rooted in Lutheranism. He did not credit himself for awakening his intellectual powers, but considered this a consequence of his Christian faith: "Religious insights gave the first impulse; only my gradually awakening religious feelings could transform my silliness, which so far had only served to stimulate my vanity, into a path towards self-knowledge … It was only then that religious convictions could take hold of me, now that I had experienced the power of religion …"[59]

His life-long Christian faith was linked to the youthful revelation in which he consciously decided to put his intellectual capacities to work – as he saw it, a decision inspired by religion. This experience also was the basis for his earnest propagation of the Christian faith later in life, although his later work in mathematics and the natural sciences – in contrast to his father's – avoided all references to God and religion.

Hermann Graßmann's worldview was a rationalistic one from the outset. It mobilized his creative potential to the point of physical exhaustion: "For I believed", so he wrote, "that intellectual talent only depended upon the energy we invest to mobilize our thoughts, – that it was not fate that shaped a human life, but that men shaped themselves and therefore also the way they were affected by their fate; I believed that there was one overriding goal for every man: perfection (being godlike), which had to be the same for everybody; – I was convinced that my phlegmatic disposition had to be completely eradicated, for I felt that it was incompatible with perfection."[60]

Starting with his confirmation (1823) and ending approximately in 1840, when he turned to mathematics once and for all, Hermann Graßmann went through a phase in which he actively fought for fundamental positions in his worldview, a struggle which also permitted him to find his place in society. He searched for meaning in life and for humanistic ideals. He grappled with himself and got to know himself by comparing his personal ideals, which had arisen from contemporary ideological tendencies, with his patterns of behavior and temperament. This struggle not only dominated his youth, but remained with him for his entire life.

On 17 September 1827, Hermann Graßmann completed his secondary-school education and received his school-leaving certificate "number 1", the best possible grade. This certificate confirms his psychological self-portrait:

"Behavior:

a) with schoolmates: His gentle, benign and helpful behavior made for a harmonious relationship with his classmates and there was never any reason for complaint.
b) with teachers: He was known for his modesty and reliability, and his teachers were extraordinarily pleased with him.

Diligence:

> He attended class regularly and attentively, thereby proving his praiseworthy and efficacious will to learn.[61] He always responded diligently to whatever task he was given."[62]

To be sure, Graßmann's final exam in mathematics showed no traces of his later originality. It was only rated second-best.

Young Graßmann was remarkably interested in music, a passion he shared with other members of the family. The ballad composer Carl Loewe, with whom the Graßmanns had shared their living quarters for an extended period of time, taught him to play piano and bass. Loewe's Romanticism probably made an early impression on Graßmann and must have influenced his general worldview.

The love of music never left him. In later years, he successfully and enthusiastically conducted the boys' choir of the Stettin "Gymnasium". He also collected Pomeranian folksongs, arranged them and sang them with his children.

After enrolling at Berlin University to study theology together with his brother Gustav, who was two years older than Hermann, a piano was Hermann Graßmann's first major acquisition. He invited fellow students who shared his gift for music to his small rooftop apartment, where they sang entire operas and oratorios to the piano.[63]

His university years began on a cheerful note, and they proved to be a new, decisive phase in the life of Hermann Graßmann. Berlin University, still a young institution, provided an intellectual atmosphere that accelerated Graßmann's development: he was becoming a bourgeois scientist.

The University, which had opened its doors in October 1810 in the context of reforms aiming to strengthen the social and political position of the bourgeoisie, was exceptionally important in the process of creating a bourgeois elite in Prussia. Wilhelm von Humboldt was the University's founder and architect. Schleiermacher and Fichte had done prior work on its programmatic guidelines. Fichte also became the University's first president and, influenced by the bourgeois reorganization of Germany, was committed to the values of bourgeois national education. Founded relatively late in history, the University of Berlin was free of traditionalistic scholastic thinking: "From the beginning, it was a modern university which represented the spirit of the ascending German bourgeoisie. It had been established under the influence of the French Revolution and German Classicism ... Wilhelm von Humboldt's ambition was to create a focal point for significant open-minded patriotic humanistic scholarship."[64] Even during the period of feudalistic rollback, the University managed to maintain a certain degree of independence and did not submit completely to the aristocratic ruling class.[65]

Hermann Graßmann's studies in Berlin, which began in 1827, were yet another important step in his personal development. He spent six semesters in Berlin. For five semesters, he focused almost exclusively on theology. He attended one of Neander's lectures, three of Schleiermacher's, one of Hengstenberg's, Strauß' and Marheineke's. Additionally, he attended lectures by F. von Raumer on the history of the German and English Reformation during his first and third semesters. He also participated in Zeune's lectures on German geography in his first semester, attended Schleiermacher's lectures on dialectics in his second, Neander's on ethics and Boeckh's on Greek antiquity in his fifth.

The church historian August Wilhelm Neander (1789 – 1850) – whose original name was David Mendel – had been a professor of theology in Berlin since 1813. Under the influence of Schleiermacher he turned his back on Judaism and became a fervent Protestant. He fought pantheism and condemned the works of D. Strauß, which were critical of traditional religious views. Written by a man disinterested in contemporary life, his work on the history of the church – parts of which had a dogmatic tone to them – was about times as remote as the 14th century. He took up a political position between Schleiermacher's liberalism and Hengstenberg's reactionary attitude. He was much admired by the leaders of Pomeranian orthodoxy. An enemy of Hegel, he attempted to show that miracles were the driving force behind the history of the church and tried to establish God as the dominating and progressive principle in history. Even though his scientific work quickly fell into oblivion, he made a strong impression on the young theologians of his time.[67]

Friedrich Daniel Ernst Schleiermacher (1768 – 1834) became the first dean of the Faculty of Theology at Berlin University in 1809. He was a progressive patriotic scholar, who, despite being reprimanded by the government for his participation in student clubs, courageously fought for bourgeois political ideals in Germany.[68]

Ernst Wilhelm Hengstenberg (1802 – 1869) was a leader of orthodox Lutheranism and became professor of theology at Berlin University in 1826. In 1842/43, he was elected dean of the Faculty of Theology. He used his influence to turn the faculty – especially in 1848/49 – into an orthodox stronghold. He vigorously fought Hegel and Schleiermacher. A leader of the alliance between old Prussian aristocracy and new Prussian orthodoxy with counter-revolutionary convictions, he aligned the church with the feudalistic rollback.[69]

The Pietist Gerhard Abraham Strauß (1786 – 1863) came to Berlin as a professor of theology in 1821. He hardly contributed anything to the development of theology. Initially a mediator between liberalism and reactionary tendencies, he later adopted Hengstenberg's position.[70]

Philipp Konrad Marheineke (1780 – 1846) had been a professor of dogmatics and church-history since 1811. He became Hegel's first follower in Berlin. Later, this created a conflict between Marheineke and his faculty, which was in the hands of orthodox Lutherans and Pietists who rejected Hegel's priority of philosophy over theology. He was university president from 1817/18 to 1831/32.[71]

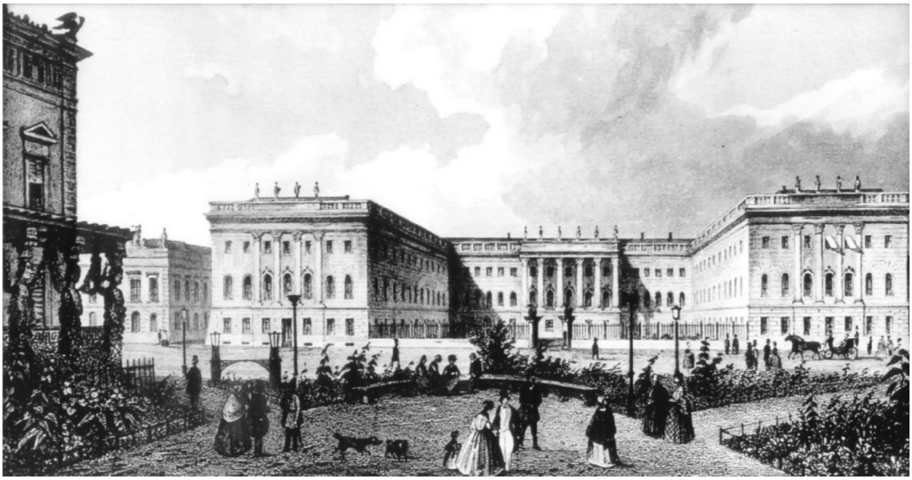


Fig. 11. Berlin University in the mid-19th century

Astonishingly, during his entire time at the University of Berlin, he never attended a single lecture on mathematics![66]

The curricula vitae are absolutely indispensable for insight into Graßmann's personal development during this period of time. We learn that Graßmann was initially strongly attracted to the lectures given by Neander. Neander, a theologian, confirmed some of Graßmann's religious convictions and laid certain doubts to rest. Graßmann's high opinion of Neander only gradually waned under the growing influence of Schleiermacher. But Graßmann still admired Neander in later years – if not so much for his teaching, all the more for his attitude and personality: "I [have] always felt profound respect and love for the truly child-like simplicity of his character and his way of lecturing. He has intense religious feelings, which help him embrace new thoughts and develop them without the slightest irritation, and he maintains a humble and modest posture", Graßmann wrote to his brother Robert in November 1836.[72] Marheinecke and Hengstenberg, in contrast, were the theologians Graßmann liked least.[73]

During his university years, Graßmann developed his independent methods of study which, at a later point in time, enabled him to approach mathematics as an autodidact. According to his professors, Graßmann was initially so impressed by academic scholarship that he believed to be drinking at the fountain of wisdom. But he quickly began to develop his own views. Step by step, he began to "follow his own paths and to understand that academic lectures can only be fruitful when enjoyed in moderation ..."[74]

The lists recording Graßmann's choice of classes show clearly that, in his last year, he turned his back on theology and became interested in philology. How can we explain this change of heart?

Fig. 12. August Boeckh (1785 – 1867)

Graßmann himself emphasized that this had not come as a sudden decision. Rather, he stated that his youthful love of country life had faded. To his dismay, rural clergymen had begun to lose interest in scientific work and Graßmann feared that the same might happen to him. Also, chances of getting a congregation in a larger town were slim. In any case, he wanted to devote as much time as possible to obtaining a universal scientific education in order to keep his interest in science alive for ever. This was to be the task of philology.[75]

He was convinced that studying philology, which at the time included many aspects of other fields of learning, would spark interests which could even survive rural isolation. Therefore, Graßmann began his studies of philology systematically, following an elaborate plan and only after having established to what extent university lectures might be useful. He aimed to begin with Greek, which was supposed to serve as a basis for Latin. Still, Latin authors had to be included in the initial program. After finding his footing in philology, Graßmann planned to focus on mathematics, which he considered too far removed from ancient languages to work on both at once.

But this extensive personal program, which was yet another manifestation of his youthful hunger for knowledge, exceeded his intellectual and physical capacities. Graßmann fell ill. Apart from the growing influence of Schleiermacher, this physical breakdown provided a second occasion to rethink his approach to life. Graßmann realized that he was carrying out his studies with an intensity that was incompatible with his temperament and that he was ruining his health. So he decided to slow down and to choose his areas of interest more carefully, without making compromises in his commitment to his work. Focusing more clearly, Graßmann decided to adopt a less monotonous work routine and a better equilibrium of work and relaxation.

In 1811 the historian Friedrich von Raumer (1781 – 1873) became a professor in Breslau. In 1819, Berlin University offered him a professorship of political science and history, which he accepted. He was elected president of the University in 1822/23 and 1842/43. In 1848 he was a center-right member of the Frankfurt National Assembly.[76]

Johann August Zeune (1778 – 1853) was one of the leading figures of the bourgeois intelligentsia at the Berlin University. During the Napoleonic occupation he was a member of the "Tugendbund", a patriotic and philanthropic organization, and one of the founding members of the so-called "Deutscher Bund". He was a follower of Pestalozzi and a close friend of Jahn, the founder of the German "Turnverein" gymnastics movement. He became a professor in Berlin in 1810. There, he spread the scientific views of A. v. Humboldt and Neo-Humanistic convictions. But, at the same time, he also showed a reactionary, nationalist attitude and became a fervent German traditionalist.[77]

August Boeckh (1785 – 1867) received a position as a professor of classical philology in 1810. During his 50 years as a scholar in Berlin, he served six terms as the dean of the Faculty of Humanities and was elected university president five times. Boeckh linked philology to historiography. He was highly respected by his colleagues and students for his scholarly achievements and for his support of liberal reforms.[78]

The historian of philosophy Heinrich Julius Ritter (1791 – 1869) had studied theology in Halle, Göttingen and Berlin (1811 to 1815). In Berlin, he became a follower and pupil of Schleiermacher. In 1817, he finished his dissertation in Halle and, after completing his habilitation, became a privately paid professor in Berlin. Only in 1824 did he receive a regular teaching position. In 1833 he left Berlin University because, even after Hegel's death, with his followers still in place, he had no chance of becoming a full professor. His philosophy was characterized by eclectic positions. "His philosophical endeavors aim to establish a Christian and theistic worldview by defending the concept of miracle and of reality in the Revelation."[79]

Graßmann's university studies taught him independence, versatility and the importance of maintaining physical and intellectual vigor. But this autodidactic approach also showed first signs of what would later reveal itself to be a form of self-isolation from the newest scientific developments. In later phases of Graßmann's work, the disadvantages of autodidactic learning appeared more strongly, modified by many other influences.

While Graßmann was changing the way he approached his studies, Schleiermacher's growing influence led to further evolutions in worldview and ethical positions. This is how Graßmann tells the story in his second curriculum vitae: "I had begun to attend Schleiermacher's seminars as early as the second semester, but I couldn't make much of what he said.[80] Instead, his sermons started to make an impression on me. But it was only in my last year that I truly began to feel attracted to Schleiermacher; and even though I was already very involved in philology, it was only then that I understood *that Schleiermacher showed*

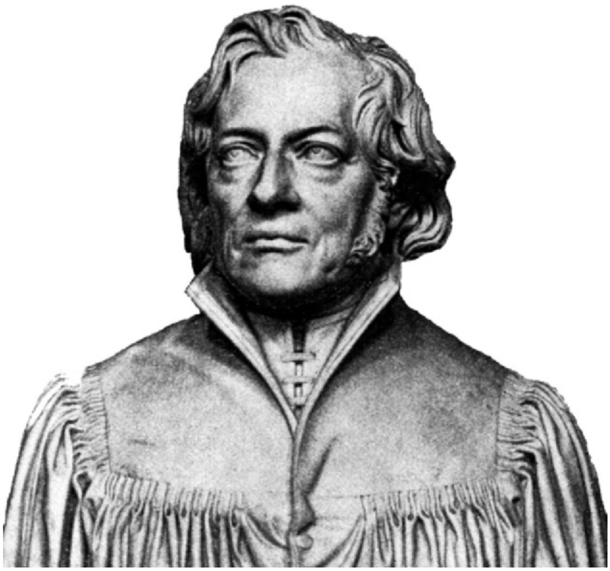


Fig. 13. Friedrich Daniel Ernst Schleiermacher (1768 – 1834)

his students how to approach every branch of science competently. He did this not by giving positive elements, but by showing how it was possible to approach every investigation correctly and continue it independently, thereby enabling the scientist to find the positive elements on his own.

At the same time, his ideas stimulated me intellectually while his sermons inspired me spiritually. This could not remain without consequences for my basic convictions and my entire way of thinking. Also, I fell ill, which gave me a certain solemnity and led me to scrutinize myself. Slowly I began to understand that my previous ideals had been wrong and incomplete. – I realized that it was useless, even harmful to fight my own temperament; useless, because I would never be able to destroy my temperament; harmful, because such a project would always harm the body and erase intellectual particularities, that is, *destroy my life as an individual* [emphasis mine – H.-J. P.]."[81]

This remarkable description of how Graßmann found the mainstays of his worldview documents Schleiermacher's double influence very clearly. Firstly, it shows how Graßmann accepted elements of Schleiermacher's ideas on epistemology and scientific methodology, which Schleiermacher mainly presented in his lectures on dialectics. And secondly, it also shows that Schleiermacher's ethics made a major impression on Hermann Graßmann.

Graßmann was willing and able to delve deeply into Schleiermacher's rich world of ideas and dialectical virtuosity.[82] In doing this, Graßmann acquired the dialectical and methodological knowledge which later enabled him to write his masterpiece, the *Extension Theory*.[83] Schleiermacher's ethics aimed at developing a consummate individual in human society by committing the individual to the society he or she was living in.[84] This prompted Graßmann to go on a long-term search for a way of life which suited his personality and to participate actively in contemporary intellectual and cultural life, especially after his return to Stettin.

Schleiermacher showed Graßmann the path towards self-awareness and an independent way of acquiring scientific knowledge. Hermann Graßmann, who considered himself a phlegmatic person, drew the following conclusion: "Therefore the phlegmatic person should not try to send his hazy reflections on an audacious flight. For the wings on which he attempts to reach the sun are not his own; therefore, like Icarus, he will quickly tumble back to earth. Instead, he must aim to make his thoughts clear and gain depth through clarity ... We must also rethink what it means to make an effort. When he makes an effort, his only goal should be to remove every external influence which

The notion of phlegm is recurrent in Graßmann's descriptions of himself. In November 1848, Graßmann wrote a letter to his fiancée in which he explained what he meant by phlegm and who the prototypical phlegmatic person was. He began by explaining the traditional four humors – phlegm, choler, melancholy and sanguine temperament –, ridding them of their negative connotations and considering them equally valuable. He wrote: "The phlegmatic personality will be the first of my four heroes of humor. But I would like to free this person from a prejudice connected to his name. You shouldn't take him for an insensitive lazybones or unemotional elephant, a person who turns his back on the world and has a hard time getting out of bed. I would rather use his honest German name and call him 'persistent'. I can easily give you a quick sketch of his personality. He knows nothing of sentimentality and impulsiveness. He finishes what he begins. If he doesn't succeed on the first try, he begins anew and perseveres until he has finished the task. He focuses on what is immediately before him. He subordinates all of his deeds and goals to the overall view of the whole, to the single idea which he has identified as his life's task and inner motivation. Therefore his entire life and work form a whole, and everything around him becomes a part of it. His will is strong, his endeavors well-planned. He pursues his goals patiently and remains absolutely true to his life's task. He has a clear and complete vision of what he is aiming to do, and the multitude of external impressions and complications endanger this vision. Therefore, he likes to isolate himself from the outside world and often seems uninterested in it, which, after all, he is not."[86]
This quotation clearly shows that Graßmann identified with this type of phlegmatic personality, the "persistent" personality, as he called it. The quotation shows that Graßmann had gone beyond the search for ethical guidelines during his years as a student, a project culminating in a positive view of his own temperament.

At the time, he had already decided on his life's goal, which was the further development of extension theory, and he had already published his *magnum opus* (A1).
After the phlegmatic, Graßmann presented the choleric as the second main type of male character – he considered sanguine ("open character") and melancholic ("deep character") essentially female personalities – and distinguished choler from phlegm: "After phlegm, choler is next on stage, his brother. Don't imagine this person to be an angry devil. I would rather call him decisive. Just like his brother, he shows not the slightest sentimentality, at least not the kind of sentimentality which keeps staring at itself, and he is equally prepared to act. But he does not aim to complete one single task, isolated from the many distractions of everyday life. Rather, these very distractions motivate his decision to act. This is his dominant trait: The decision to act arises from the unexpected turmoil of everyday life. He acts abruptly, as if some secret power were guiding him. He decides quickly and doesn't need to give it much thought. Instinctively, he takes whatever measures he must to reach his goal. People look to the decisive person to shape the form of their lives, or change the course of history. The phlegmatic personality, in contrast, puts more and more distance between himself and everyday life and, using his impartial judgment, is predestined to make long-term plans or to construct scientific edifices."[87]
According to Graßmann, the phlegmatic personality was the researcher, calmly exploring his scientific ideas, while the choleric (or decisive) personality – in many ways similar to Graßmann's brother Robert – was the active politician.
Nevertheless, Hermann Graßmann did not believe that any of these clearly delineated types represented the ideal human character. True humanity could only arise when they all came together. But, as Schleiermacher also thought, the ideal of true humanity was accessible only to humanity as a whole, whereas the individual represented the limitations of what humanity could be.

clouds his mind. And I strove to do so as best I could in all aspects of life (not just in science)."[85]

The wide range of knowledge which Graßmann acquired during his university years formed the essential basis for further developments in his worldview and for his prolific achievements in different branches of science. Especially his studies of philology came to fruition in his later philological work, which received much more public recognition than his brilliant long-term endeavors in mathematics.

Graßmann, as we have already seen, did not attend any lectures on mathematics during his time as a student in Berlin. Even though he emphasized in his curriculum vitae that, at the time, he had already been interested in mathematics, he did not give any specifics. Probably, he merely scratched the surface.

We may assume, though, that Hermann Graßmann[88] carefully read and studied his father's mathematical works, which were published in this period of time: *On the Concept and Extent of the Pure Theory of Number* ("Über den Begriff und Umfang der reinen

Zahlenlehre", ZL) and the treatise *On Physical Crystallonomy and the Geometric Theory of Combinations* ("Zur physischen Krystallonomie und geometrischen Combinationslehre", KRY). Given the fact that his curriculum vitae gives many details, but says nothing about the subject of his mathematical studies in university days it is safe to assume that Graßmann did not work on mathematics systematically. We may also assume that, when Hermann Graßmann became more deeply involved in mathematics at a later point, these two treatises had influenced him strongly and that his father's ideas were still with him.[89]

1.4 The path to independent mathematical achievements (1830 – 1840)

Hermann Graßmann returned to Stettin in the autumn of 1830. His immediate goal was to prepare the exams he needed to become a secondary-school teacher. His health was good, and he went back to his initial plan of independent, autodidactic studies. From autumn 1830 to December 1831, he focused on mathematics, which he combined with physics (relying on a textbook by Ernst Gottfried Fischer 1826) and the natural sciences (zoology, botany and mineralogy, using material by Georg August Goldfuß). For a change of pace, he worked on philological, mythological and historical problems. At the same time, he began to study geometry with Adrien-Marie Legendre's *Eléments de Géométrie* (1823) and arithmetic, inspired by Georg Freiherr von Vega (1802). As he remarked in his curriculum vitae of 1833, he studied geometrical and arithmetical theories simultaneously because it made the work less monotonous. Geometry, he explained, "graphically shows what it has proven or aims to prove, whereas arithmetical theories must be grasped intellectually and in thought."[90] Next was the theory of combinations, "which could show him the path to more sophisticated fields of mathematics"[91], and spherical trigonometry, which he studied using his father's books. He then went on to work on cone sections, following Friedrich Wilhelm Schneider (1834), which he used to approach differential and integral calculus. Graßmann also studied the theory of finite and infinite series and of algebraic equations of the third and fourth degree, and finally differential calculus as it was presented by Johann Tobias Meyer (1819).

Graßmann's study method was the ground on which the brilliant ideas set down in his *Extension Theory* could grow. By combining geometry, arithmetic and the theory of combinations, Graßmann sharpened his eye for general patterns and began to think in terms of analogy. Graßmann confirmed this himself when he claimed that he had discovered geometrical addition and multiplication of displacements (vector addition and the exterior product of vectors) as early as 1832[92], thereby leaving Möbius far behind and anticipating the key ideas in his *Extension Theory*.[93]

For the time being, Graßmann's insights were fragmentary, mere by-products of the work he was doing to become a teacher.

Around Easter 1831, Hermann Graßmann was admitted to the Stettin teachers' seminar, which was closely connected to the local "Gymnasium". Graßmann had to declare that, after completing his studies, he would serve as a teacher for at least three years or pay back half of his stipend of 150 taler per year. He also was obliged to teach a given number of hours at the local secondary school.[94]

During the summer of 1831, Hermann Graßmann worked as an assistant teacher at the Stettin "Gymnasium" and taught German ten hours a week and geometry two. At the same time, he was preparing three graduation theses for the examination commission in Berlin. Among these papers was a work on mathematics entitled *On the Concept of the Differential* ("Über den Begriff eines Differenzial"). Friedrich Engel made the following comments on the paper: "It contains general, sometimes slightly philosophical reflections on the concept of infinity, on infinitely small magnitudes and differentials. Obviously, Graßmann could not be original here, but it is clear that he had thought about these concepts thoroughly and coherently, explaining them very well ..."[95] Even though Engel was reserved in his judgment, he nevertheless recognized that Hermann Graßmann had used a personal approach (which was inspired by his father). In this approach mathematical thoughts were always accompanied or even initiated by philosophical reflections.

The oral examination took place on 17 December 1831. Graßmann's diploma is dated 31 December, and it gave him permission to teach philology in lower and intermediate grades, history in eighth grade, mathematics in tenth grade, natural sciences and German in lower and intermediate grades as well. The only thing the examination committee expressed doubts about was his philosophical knowledge. On the diploma, the committee said: "The examination in philosophy ... showed that the candidate possesses the kind of philosophical knowledge needed to methodologically develop general concepts, but he lacks knowledge about the results of philosophical research as such ... Therefore he is not yet capable of teaching philosophical propaedeutics and conducting exercises which aim to stimulate and elaborate scientific thinking in general."[96]

This confirms statements of Graßmann's in which he explained that, during his university studies, he was mainly interested in *philosophical methodology*, but not in *philosophical systems* and their content. Of course, his later scientific achievements would prove that he was very much able to contribute to "scientific thinking in general".

After having passed his examinations, he remained an assistant teacher at the "Gymnasium" until autumn of 1834. He signed up for military service in April 1832, but never became a soldier due to his "generally weak physical condition"[97] and joined the reserve. Immediately after the examination he was mostly busy teaching and kept himself occupied with botanical studies and barometrical measurements. In Greek literature, he focused on Plato's dialogues. He also continued his mathematical research.

In 1834, Graßmann passed his first examination in theology. He was still thinking about a church career, according to his brother.[98] But, apparently, he had by then become quite keen on doing scientific work. When in October 1834 he had the opportunity to become a teacher of mathematics at the Berlin School of Commerce, he immediately took the job.

J. Steiner had been granted a position at the University of Berlin, and this had created a job opening for a teacher of mathematics at the School of Commerce. Steiner, the extravagant rediscoverer of synthetic geometry in Germany, had been teaching at the Berlin School of Commerce since 1829 and was happy to leave the "detestable grind of school service" behind.[99] Looking to replace Steiner, the administrators in charge chose Graßmann, "who, judging by his diplomas and other information we have been able to gather, seemed to be better qualified than all other applicants ..."[100] In a report to the administrative committee, the director of the school gave the following description of Graßmann: "*Herr* Graßmann is a learned young man. Also, he obviously has spent much of his time thinking about the elements of mathematics, and he masters them very well. But he seems to have led a solitary life and therefore has difficulty in regular social life. On such occasions, he is withdrawn, timid, quite sheepish and, as a consequence, behaves clumsily."[101]

Graßmann started out teaching 10 hours a week in the lower grades, while Steiner volunteered to continue higher-level classes for another six months. Also, the school obliged Steiner "to help the school and the teachers of mathematics working in lower grades by attending, as long as your time will permit, the classes these teachers give in higher grades, by explaining the essentials of your method to the teachers, by answering their questions and, if possible, training them to use your method"[102]. Notwithstanding the school's agreement with Steiner, he did not have any noteworthy personal influence on Graßmann. It is unlikely that they developed closer ties.

The differences in character alone were too great. Graßmann, twenty-five years old, was timid and withdrawn; thirty-eight year-old Steiner was "known, even famous for his vulgarity and rudeness ..."[103] Steiner's name never appears in the letters Graßmann wrote to his family during this period. Still, his brother Robert later said that Hermann "was now deeply involved with mathematics, a project in which the work of Jakob Steiner, his predecessor on the job, had much to offer him. But Steiner's path was one of synthetic construction, whereas Graßmann's was one of analytical derivation."[104] Engel was certainly correct when, in this context, he stated: "But it is evident that, though Graßmann studied Steiner's works, these had no noteworthy impact on his way of approaching mathematics."[105]

Graßmann's very hesitant and "apprehensive"[106] corrections in the manuscript of his father's mathematical textbook (J. Graßmann 1835), which at the time was being published by Reimer, show his lack of confidence in his own mathematical capabilities.

Fig. 14. Jakob Steiner (1796 – 1863)

Graßmann's emotional condition was shaky and, at the beginning of his second stay in Berlin, clouded by the death of his youngest sister, who had died in February 1834 at age three. Also, far away from friends and relatives, he was experiencing big-city loneliness. His first letters were filled with religious ruminations and bitter thoughts about the world. His timid and passive personality played a part in his escape to self-contemplation. In a letter from January 1835, Graßmann told his father: "My views on everything concerning the practical aspects of life are still insecure. They change almost on a daily basis ..."[107] But, as Graßmann hinted in his curriculum vitae, he now had to stand on his own two feet and rely on himself. It was during this period that his worldview and his religious convictions gradually lost their mystical elements. Apart from religion, he increasingly began to appreciate science as a form of *personal experience* and his interest in pastoral work slowly disappeared. In March 1835, he wrote a letter to his father in which he described this personal development: "Recently I have felt much calmer and, in my religious convictions, I have begun to make my peace with everyday life. It has become clearer to me that it is useless to try and experience God *directly*. In fact, such an attitude can even cloud our capacity for feeling the divine because all kinds of emotions get mixed up in it. I am starting to see that, as in everyday life, God can be experienced *indirectly* in science. Here we can also fulfill God's will, and we can also strive for kingdom come. I feel that my senses are getting clear again and that I can now experience God indirectly from a different perspective – a more spiritual and truthful perspective. And I also believe that it is not a sin when somebody is so committed to science that he forgets himself in his search for truth, that is, a higher state of existence ..."[108]

At a later point in time, this pantheistic attitude became more pronounced in Graßmann's scientific creativity. Even though he assumed scientific work to be a "purely spiritual affair ... with God as its final goal"[109], the concept of God does not appear explicitly or implicitly in any of his scientific treatises, which, in this respect, are different from his father's works. God came before or beyond science, but He was not a part of science.

Once Graßmann had gained an optimistic perspective on science, he also gradually adopted Schleiermacher's ethics and gradually freed himself from his isolation: "I am slowly realizing that social life, with close friends or larger groups of people, is more than mere recreation. It is an occasion for learning; in fact, it is our sacred duty. All my attempts to limit my activities to the simple occupation of teaching have failed, and had to fail, for I must participate in all aspects of social life. This is the only way to escape scholarly small-mindedness; it is the only way to become a voice and the only possible way to experience inner growth."[110]

This was a first step towards what would later become Graßmann's involvement in social activities, such as his participation in numerous clubs and organizations and his contribution to the scientific and cultural life of Stettin. But initially Graßmann was content with becoming a member of the Stettin Freemasons, where many of the town's most respected citizens met, among them his father.

Berlin turned out to be an intermezzo for Graßmann. When in late 1835 he was offered a job at the newly established Stettin "Bürgerschule", he immediately canceled his contract at the School of Commerce and returned to Stettin on 1 January 1836. He would never leave Stettin again.

A letter to his brother Robert written in late February 1836 shows how Graßmann regained his good spirits in Stettin, but also how highly he valued his days in Berlin. He wrote: "I have been in Stettin for one and a half months now, and I can't even begin to tell you how much I enjoy being here, compared to living in Berlin. Naturally, when I left for Berlin, I knew that I would have to do without many of the things that make life comfortable and enjoyable, but I also tried to see that my new situation, my new position and the self-reliance I would definitely have to show, along with the intellectual stimulation I was likely to receive in Berlin which could mobilize all of my intellectual and moral powers, would in any case further my development. And I was right. I do not regret having lived there and I don't feel as if I should have continued my old life instead. But it is true that lately I had become quite weary and tired of a chaotic life which lacked all loving and happy commitments. So I was very glad to be back in Stettin. Here the flow of inspiration is gentler, but it touches me more. The circle of people I work with is smaller, but my influence on the people I

Fig. 15. Hermann Graßmann as a teacher.
Pencil drawing by a pupil

meet is stronger, as is their influence on me. I have less scientific material at my disposal, but I can use it more efficiently."[111]

On 5 January 1836, Graßmann petitioned the Stettin city council "for a transfer to the newly created position at our 'Ottoschule'".[112] This newly founded school aimed to provide a better education than what the "Bürgerschule" had to offer, but still below secondary-school level. The school's first headmaster was Christian Heß, Hermann Graßmann's brother-in-law. Heß undoubtedly supported H. Graßmann's application for the position of the so-called second teacher of science. Graßmann began teaching immediately, even though he only officially got the job in March 1837.[113]

He taught 24 hours a week in the highest and second-highest grades. Of these 24 hours, four were mathematics, three physics. We should note that Graßmann carried out experiments in his classes and that he encouraged his pupils to continue these experiments at home. "My only current occupation is work related to school, and I am approaching this work as if my position at school were permanent", Graßmann told his brother in 1836, "for, again, the only way to deepen my scientific insights is to grasp the subject matter as thoroughly as possible. I even believe that I am now on the best possible path because its direction is determined by my official obligations, but this does nothing to restrain the intensity of my scientific ambitions."[114] Graßmann had begun to follow in his father's footsteps. In preparing the material he was to teach and analyzing seemingly elementary concepts of mathematics and physics, he gained the good judgment and the critical attitude which later enabled him to completely reconstruct geometry and formulate the general structure of arithmetic in a uniquely precise way, setting an example and going far beyond what M. Ohm had to offer.[115]

In 1839, Graßmann produced a treatise called *The Derivation of Crystalline Structures from the General Law of Crystallization* ("Ableitung der Krystallgestalten aus dem allgemeinen Gesetze der Krystallbildung"). This treatise was the first result of his "classroom" methods, a text which took up a topic from the curriculum of the "Ottoschule" and continued his father's *On Physical Crystallonomy and the Geometric Theory of Combinations* ("Zur physischen Krystallonomie und geometrischen Combinationslehre", 1829).[116] Taking the principle of symmetry as a point of departure, the treatise was about deriving simple crystalline structures and categorizing them systematically. Graßmann limited himself to elementary geometrical relationships. The treatise, which Graßmann had intended to use in class, caught Möbius' attention. Möbius had spent some time working on similar problems. In 1854, he wrote a letter to Graßmann: "I read this treatise with great pleasure. I can only hope that in your new version of the *Extension Theory* you will explicate your theories just as clearly as you did here."[117]

It seems as if his father's influence grew during this period of time. In October 1836, Hermann Graßmann became a member of his father's Physics Society. The society consisted of 20 members, all local citizens, and Hermann Graßmann's lectures contributed to the scientific quality of the proceedings.

But he still had not completely given up on the thought of working for the church. He was still undecided whether to pursue his mathematical and scientific interests or focus on theology. For all this time, he had continued to work on theological problems. Even though he was very busy teaching, he nevertheless signed up for the second examination in theology in May 1838. This meant that within a year he would have to hand in three theological treatises. On the tenth of July, Graßmann took oral and written examinations and two days later he received his diploma: "very good, passed and eligible"[118]. It is highly remarkable that Graßmann managed to obtain such an excellent result despite his ongoing obligations. Astonishingly, Graßmann had asked to repeat his examinations in mathematics and physics 4 ½ months before taking the examination in theology. On 28 February 1839, he had written to the chairman of the scientific examination commission in Berlin: "I ask you very kindly to admit me to a renewed examination of my knowledge in physics and mathematics. Since taking my first examination eight and a half years ago, receiving the present diploma, I have continued working on these two subjects. I am about to apply for a teaching position in mathematics and physics[119], but I cannot hope to succeed without prior confirmation of my qualification: therefore I have found it necessary to bother the high commission with my humble request, asking it to examine my knowledge in physics and mathematics a second time."[120]

Professor Conrad, who was in charge of mathematics in the commission, received Graßmann's letter on 4 March 1839. That same day, Conrad forwarded it to the commission, noting: "*Assignment for mathematics and physics*: the theory of low and high tides"[121].

The assignment was sent to Graßmann on 10 March, and he returned it on 20 April of the following year, along with the following request: "Since I need the results of the examination for a job application, I kindly ask you ... to begin as soon as possible with the remaining procedures of the examination."[122] The oral examination took place on 1 May. Graßmann obtained very good results. In its assessment, the commission concluded that he was "perfectly capable of teaching mathematics, physics, mineralogy and chemistry in any position at a secondary school."[123] Concerning Graßmann's competence in mathematics, we find the following passage: "The candidate has thorough and extensive knowledge of mathematics, of the entire field, including mechanics and higher analysis. In his examination thesis, he treats the theory of high and low tides in detail and coherently, even relying successfully on an unusual method which deviates in some points from the theory of Laplace. During the oral examination he quickly and calmly grasped the problem he was given and showed such a high degree of competence that we declare him completely capable of teaching mathematics in all grades of 'Gymnasium'. "[124]

Graßmann passed his examination in theology and the second round of qualifications in mathematics and physics, showing an admirable and exceptional commitment to the task. This not only brought a preliminary end to a period of intense intellectual strain, but it also represented a definite turn towards mathematics. Graßmann's approach to the theory of high and low tides, which Engel considered as important as the thesis written by the young Weierstraß[125], finally gave Graßmann confidence in his mathematical capabilities and made his turn towards mathematics definite.

Even though the paper's corrector, C. L. Conrad, did not at all appreciate the relevance of Graßmann's groundbreaking mathematical insights and simply mentioned his "unusual method"[126], Graßmann himself knew that he had made an extremely important discovery. He would devote the next twenty years of his life to developing and propagating his ideas.

1.5 Mathematical productivity and the struggle for recognition (1840 – 1848)

In 1840, Graßmann's work on the problem of high and low tides opened a new chapter in the story of his personal scientific development. Once and for all, he concentrated on mathematics and gave up on the idea of working for the church. In May of 1847, Graßmann sent a letter to the Prussian Minister of Culture and Education, Eichhorn, in which he applied for a teaching position and described his change of heart: "Initially a theologian and very attracted to theology, I nevertheless have always been very interested in mathematical problems. While I was still trying to decide whether to choose the one or the other, studying both fields at the same time, I made a remarkable series of discoveries and suddenly became involved in a completely new aspect of mathematics. ... I was unexpectedly confronted with a completely new mathematical method which often made it surprisingly easy to solve the most difficult problems of mathematics and physics. But I was still at the beginning. ... I decided that it had to be one of my life's tasks to work in this scientific field."127

Graßmann made a similar point in his treatise *On the Loss of Faith* ("Über den Abfall vom Glauben", 1878): Only "by making discoveries in the field of mathematics", he claimed, had it "become certain" that his "life's task lay in science". It was only at that point that he had given up on the "practical goals" connected to theology.[128] In 1839 Graßmann had taught himself the essentials of analytical mechanics, higher infinitesimal calculus and the calculus of variations on the basis of works by Lacroix and Lagrange.[129] In his work on the theory of tides, Graßmann had had to master Lagrange's *Mécanique analytique* (1811/15) and Laplace's theory of high and low tides, which Laplace had developed in his most important work, the *Traité de mécanique céleste* (1799 – 1827). Confronted with these highly complex theories, Graßmann returned to his ideas on vector calculus from 1832[130], hoping they would help him understand these theories. In his graduation thesis, Graßmann showed great competence in Laplace's "obscure and difficult" theory. But even though this alone would have fulfilled the commission's expectations, already making the candidate's scientific future seem very promising indeed, Graßmann went on to develop completely new mathematical tools – vector calculus and external algebra – in order to give a better mathematical representation of the theory of tides. He aimed to continue work on this new mathematical method, the relevance and efficiency of which he saw clearly.[131]

J. W. Gibbs was very much in favor of including the paper on the theory of high and low tides in Graßmann's collected works, which were published after his death.[132] The paper offers interesting insights on how Graßmann discovered and developed many ideas related to his new method. Graßmann was in the difficult position of having not only to treat the theory of high and low tides but of having to explain, in context and step by step, the elements of his new method. Therefore, he constantly switched back and forth between presenting the consequences and structure of his new method and his results in the theory of high and low tides. Given the limitations of the paper – it already exceeded the number of pages of treatises of this type – Graßmann could not offer a proper foundation for his method and in many cases had to rely on analogies. In this sense, his examination thesis is a precursor of *Extension Theory*, which offered a sophisticated and systematical form of presentation.

Hermann Graßmann's personal library contained the following books on mathematics:

- Lagrange, *Théorie des fonctions analytiques* (1797) and *Mécanique analytique*;
- Poncelet, *Traité des propriétés projectives des figures* (1822);
- Moigno, *Leçons de calcul différentiel et de calcul intégral, rédigées d'après les méthodes de A.-L. Cauchy* (1840/61);
- Verhulst, *Traité élémentaire des fonctions elliptiques: ouvrage destiné à faire suite aux traités élémentaires de calcul intégral* (1841);
- Ohm, *Versuch eines vollkommen consequenten Systems der Mathematik* (1829 – 55);
- Magnus, *Sammlung von Aufgaben und Lehrsätzen aus der analytischen Geometrie* (1833, 1837);
- Steiner, *Systematische Entwickelung der Abhängigkeit der geometrischen Gestalten voneinander* (1832), and also *Die geometrischen Konstruktionen, ausgeführt mittelst der geraden Linie und Eines festen Kreises* (1833);
- Möbius, *Der barycentrische Calcul, …* (1827) and his *Die Elemente der Mechanik des Himmels* (1887a).[134]

It is unfortunate that Professor Carl Ludwig Conrad, charged with assessing Graßmann's thesis, was completely incapable of recognizing its value. Conrad was a teacher of mathematics and French at the "Königliches Joachimsthalsches Gymnasium" in Berlin and never published any work of his own.[133] This misjudgment was a first symptom of what would become a problem for Graßmann, namely that, for most of his life, his achievements went unnoticed and unappreciated. In mathematics, Graßmann was mainly an autodidact. He belonged to no particular mathematical school or movement and had no "great master". His personal library included only a small number of books on mathematics.

His father had been his only teacher and source of inspiration. Also, Graßmann shared his father's and his brother's taste for replacing traditional mathematical terminology with an extravagant German vocabulary. For these reasons Graßmann's writings were extremely hard to understand.

Nevertheless, in his theory of the tides, Graßmann brilliantly demonstrated how important his new theory was for problems of complex mechanics. Regrettably, Graßmann never found the time to revise the paper for publication, as he had planned. He also had to postpone the systematic development of his mathematical ideas: his obligations as a teacher took up more and more of his time.

Graßmann did not obtain the position he had mentioned in his letter to the commission. Even though the "Friedrich-Wilhelmsschule" had in fact been founded, with

Victor Schlegel on Hermann and Robert Graßmann's work on the theory of language:
"The *Dialectic*, by his admired master Schleiermacher, had just been published, and it attracted him strongly, briefly sending him in a different direction, which he began to work on with his brother Robert. They spent the following year (1841) working on a philosophical theory of language, the results of which he published in an *Outline of German Grammar* and the *Guidelines for Teaching German*. (Both books, the latter a collaboration with his brother, were published in 1842)."[136]

Graßmann's brother-in-law C. Scheibert as its headmaster, he remained at the "Ottoschule" until 1842.

In March 1842, Graßmann used the "Ottoschule" as a platform to publish an *Outline of German Grammar* ("Grundriß der deutschen Sprachlehre"). This text is especially interesting because of its methodological structure. It foreshadows the importance of Schleiermacher's *Dialectic* for the *Extension Theory* and its general concept of science.[135]

Graßmann's classes covered a wide range of scientific and mathematical topics, including zoology and mineralogy, physics and chemistry. He considered experiments an especially important aspect of his pedagogical work.[137] Almost daily, he and his higher-level students carried out chemistry and physics experiments in a special classroom, constructing their simple instruments themselves.[138] Thus Graßmann was not a withdrawn mathematician with a philosophical vein. Graßmann's work draws its special magic from his unique way of combining very abstract mathematical developments and an excellent approach to experimental methods.

In 1842, Graßmann's professional situation changed. Stettin city council decided to grant him the last remaining permanent position at school. But this was not for long: at Easter 1843, Graßmann was given official permission to switch positions with a teacher at the "Friedrich-Wilhelmsschule", where he remained until his father's death (1852). In the course of his new teaching responsibilities, he composed a number of smaller texts for German and Latin classes.[139] Almost in passing, between Easter 1842 and autumn 1843, he wrote his mathematical masterpiece, *Extension Theory* (A1).

This book presented a completely new approach to geometry. The reader had to digest extensive philosophical introductions, an abstract theory of conjunctions which was to serve as a basis for the entire field of mathematics, the almost complete absence of mathematical formulas, the rejection of geometry as a mathematical discipline and an n-dimensional theory of mathematical manifolds which did not rely on metrics.[140]

Extension Theory was in bookstores by 1844 – and hardly anybody noticed. Graßmann's methods and results were unheard of, sounded peculiar and lay completely outside the mainstream of mathematical research on the European continent. His conclusions were hard to understand. Scholars took no notice of the book.

Moritz Cantor (and August Leskien) give a clear summary of the reasons why *Extension Theory* remained insignificant at the time: "The book [the *Extension Theory* of 1844 – H.-J. P.] is a first expression of what, after Riemann, has been designated as manifolds: a theory of functions, geometrical in its outer appearance and with geometrical designations which only in special cases correspond to geometrical representations. But since it uses geometrical designations for concepts which have nothing to do with space as we experience it, these concepts are submitted to operations which, up to this point, had never been part of geometrical proofs or constructions. The reader is told that the magnitude of a straight line which intersects two points is the same as the product of the two points, that the surface-area and position of the plane of a triangle situated between three points is the same as the product of these three points: this scared readers away, especially since – at the time – the author's name was unknown and readers were thus hardly inclined to question their own competence."[141]

The only mathematician in Germany who could appreciate Graßmann's ideas was A. F. Möbius. In the years between the publication of Möbius' *Barycentric Calculus*[142] and *Extension Theory*, Graßmann had discovered that both projects had points in common. In 1844, he visited Möbius in Leipzig and, on 10 October 1844, asked Möbius for


Die Ausdehnungslehre von 1844

oder

Die lineale Ausdehnungslehre

ein

neuer Zweig der Mathematik

dargestellt

und

durch Anwendungen auf die übrigen Zweige der Mathematik,

wie auch

auf die Statik, Mechanik, die Lehre vom Magnetismus und die Krystallonomie erläutert

von

Hermann Grassmann.

Zweite, im Text unveränderte Auflage.

Mit 1 Tafel.

Leipzig
Verlag von Otto Wigand.
1878.

Fig. 16. Title page of the second printing of the *Extension Theory* of 1844

a review: "... I am convinced that, since nobody is closer to the ideas presented there [in *Extension Theory* – H.-J. P.] than you are, only you are capable of judging the book properly ..."[143] Möbius gave a hesitant response. He wrote that he was happy to have found a like-minded thinker, "but our like-mindedness remains confined to questions of mathematics, not of philosophy. ... I am therefore incapable of judging, or even understanding, the philosophical dimension of your excellent book. In your book, the philosophical dimension is more fundamental than mathematics. I realized this after attempting more than once to study your book *uno tenore*. Its high level of philosophical abstraction always stopped me."[144]

Möbius was not unwilling to write the review. He was just being frank about his inability to do justice to Graßmann's book. Möbius' correspondence with Baltzer and Apelt, whom Möbius asked for their opinion on *Extension Theory*, proves his good will. Möbius was also perfectly willing to ask Drobisch for a review. But everybody had trouble understanding the book. Möbius disliked the high level of philosophical abstraction in the *Extension Theory*. Apelt thought it too abstract, contrary to mathematical common sense and completely lacking in intuitive clarity. According to Apelt, "concepts are not the source of mathematical knowledge, but intuition"[145]. When Baltzer studied the book, it simply made him feel dizzy and disoriented.[146] Not even Gauß and Grunert, who received a copy from Graßmann as a gift, could grasp the "tendency" behind the book. Gauß said that he had no time to "familiarize"[147] himself with Graßmann's terminology. Grunert remarked that he had been unable to divine where the book was headed: "I also would have hoped", he told Graßmann in a letter, "that you would have refrained from getting so involved in philosophical reflections and that, instead, you would have stuck to your original plan of using the Euclidean form."[148]

On 14 December 1844, Gauß sent the following letter to Graßmann thanking Graßmann for sending him *Extension Theory*:
"...caught up in a muddle of different tasks, I have looked at your book and I think I realized that its tendencies are *somewhat similar* to the paths I have been wandering on for almost half a century. Only a small part of this research was published in 1831 in the "Comment" of the Göttinger Societät and a little more, in passing, in the Göttingsche Gelehrte Anzeigen (1831, number 64). It is the essence of a *metaphysics* of complex magnitudes, of which I have spoken in many of my lectures but of which only samples, which can only be identified *as such* by a well-trained eye, have been published elsewhere. Nevertheless, there seems to be only a distant and partial similarity between our tendencies. I am aware that if I really wanted to grasp the essence of your book, I would first have to familiarize myself with your peculiar terminology. But since I would have to free myself from other occupations, I do not want to wait any longer in thanking you very kindly for sending me your book..."[149]

In his treatise *New Theory of Electrodynamics* (1845a) of 1845, Graßmann criticized Ampère's law of magnetic forces between infinitesimal current elements. Ampère had found this law by observing how closed currents interacted, referring to infinitesimal partial currents. He had supplemented these observations with the hypothesis that the direction of the interaction of current elements (as in the case of the gravitational interaction of point-masses) coincided with the connecting line between their centers. Graßmann criticized this additional hypothesis. His research into vectors had led Graßmann to doubt that vectorial magnitudes such as current elements interacted as Ampère had assumed.[150] Graßmann also disliked the complexity of Ampère's law:

$$dF = -\frac{I_1 \,.\, I_2 \,.\, dl_1 \,.\, dl_2}{r^2} \,.\, (\cos(dl_1 \,.\, dl_2) - 3/2\cos(dl_1 \,.\, r)\cos\,(dl_2 \,.\, r)$$

dl_1, dl_2 ... conductor elements,
dF ... force between dl_1 and dl_2,
I_1, I_2 ... currents through dl_1 and dl_2,
r ... distance between dl_1 and dl_2.

Graßmann sought for a new, vectorial way of treating this elementary law. By introducing the term "angle-current", Graßmann found the following formulation of this law, an inversion of the law of Biot and Savart:

$$dF_{12} = k \,.\, \frac{I_1 \,.\, I_2}{r_{12}^3} \,.\, [dl_2 \,.\, [dl_1 \,.\, r_{12}]]$$

k ... constant,
dl_1, dl_2 ... conductor elements in the direction of the currents,
$I_1, I_2,\ dF_{12}$ ... force of dl_1 on dl_2,
r_{12} ... vector pointing from dl_1 to dl_2.

When closed currents are integrated, Ampère's law and Graßmann's "elementary law" both lead to the same result, namely:

$dF = [I \,.\, dlB]$ (also known as Ampère's law).

What is remarkable about Graßmann's "elementary law" is that it contradicts the Newtonian principle of the equality of action and reaction, if one takes the action at distance theory of electrodynamics into account, which would have been the case in Graßmann's time. If one takes the field action theory into account, which came up later, these contradictions disappear. Even though Graßmann was not aware of it, his vectorial perspective already hinted at later developments in electrodynamics.
For an extended period of time, the search for an experimental criterion favouring or contradicting Graßmann's and Ampère's "elementary laws" stimulated research in this field. In modern scientific literature, Graßmann's law still exists as a useful mathematical tool, even though it is sometimes erroneously designated as "Ampère's law".[151]

After Grunert had hinted at this possibility, Graßmann was left with no other option but to review the book himself. His *Short Introduction to the Meaning of Extension Theory* ("Kurze Übersicht über das Wesen der Ausdehnungslehre"), which was published in "Grunerts Archiv" in 1845, remained the only review of *Extension Theory* in scientific literature – the public ignored it.

Clearly, the book was not going to be a success. But Hermann Graßmann began to propagate his ideas by publishing contributions to interesting scientific problems. 1845 saw the publication of his *New Theory of Electrodynamics* ("Neue Theorie der Elektrodynamik") in a journal called "Poggendorf's Annalen". In this article, he criticized Ampère's law for magnetic forces between two small conducting elements and used his geometric calculus to simplify it elegantly.

Surprisingly, the important physicist Clausius, starting from a completely different point of departure, reached the same conclusion 30 years later. Clausius, who had been misinformed about the content of Graßmann's work, immediately recognized Graßmann's achievements publicly once Graßmann had explained them to him. In a letter to Graßmann, he wrote: "I can tell you sincerely that I am especially pleased to have had the good fortune of encountering you, the son of my beloved and admired teacher, in our scientific work."[152]

Apart from this article on mathematical physics, Graßmann began to publish a number of texts on the theory of algebraic curves in 1846. In these publications, he elaborated on a theorem about the generation of algebraic curves of arbitrary order which he had established in passing in *Extension Theory*.[153] His method of generating these curves was impressively simple. Möbius was enthusiastic about Graßmann's work: "The articles on how to generate algebraic curves with lines which turn around fixed points which ... you have published in Crelles Journal have captivated me completely. I marvel at the extreme simplicity of the symbolism you use."[154] 25 years later, when F. Klein discovered these investigations, he praised them highly in his *Development of Mathematics in the 19th Century*: "The so-called 'linear constructions' or the generation of algebraic structures with 'planimetric products' deserve special attention. This is a theory which unfortunately is quite unknown despite the fact that it is especially easy to grasp"[155]. And Klein went on to say that, with this theory, Graßmann had managed to "create the simplest foundation for the theory of algebraic curves".[156]

However, with only a few exceptions, Graßmann's contemporaries failed to appreciate the significance of his theory.[157]

Graßmann's first, and – for a long time – last, public success was his reply to a prize question which the Jablonowskian Society of Science in Leipzig had presented to the public. Möbius had informed Graßmann about the call for papers. Möbius saw the connection between *Extension Theory* and the prize question, and he believed that this was an opportunity for Graßmann to elaborate on his ideas. In a letter of

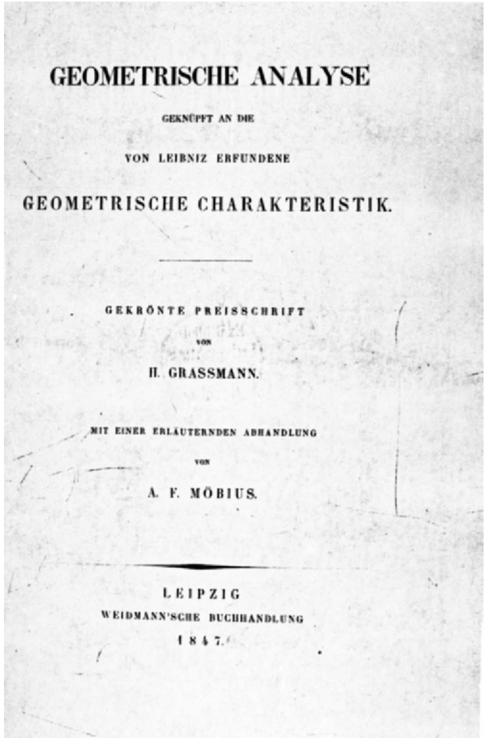
The **prize question of the Princely Jablonowskian Society of Science** in Leipzig, which had been posed to commemorate Leibniz's 200th birthday, ran as follows:
"We still possess some fragments of a geometrical characteristic developed by Leibniz ... in which the respective positions of geometrical loci are determined directly by symbols without recurring to the size of lines and angles. This characteristic thus differs from our algebraic and analytical geometry. The question is whether this calculus can be reconstructed and further elaborated, or whether we can find a similar approach which seems equally feasible..."[159]
This fragment of a "geometrical characteristic", which Leibniz had sent to Huygens on 8 September 1679, is reprinted in Graßmann's collected works (in French).[160]

2 February 1845, Möbius wrote: "Finally, permit me to send you a folder from our Jablonowskian Society of Science, in case you might want to respond to its mathematical prize question."[158]

The prize question referred to fragments of the Leibnizian geometrical characteristic which had first been published in 1833. In these fragments, Leibniz had developed ideas which came very close to what Graßmann was working on.[161]

It is one of the greatest coincidences in history that, in early 1844, Drobisch had posed a prize question – Graßmann's *Extension Theory* was still unpublished – for which only Graßmann could provide an answer. He also remained the only competitor for the prize.

Fig. 17. Gottfried Wilhelm Leibniz (1646 – 1716)


GEOMETRISCHE ANALYSE

GEKNÜPFT AN DIE

VON LEIBNIZ ERFUNDENE

GEOMETRISCHE CHARAKTERISTIK.

GEKRÖNTE PREISSCHRIFT

VON

H. GRASSMANN.

MIT EINER ERLÄUTERNDEN ABHANDLUNG

VON

A. F. MÖBIUS.

LEIPZIG

WEIDMANN'SCHE BUCHHANDLUNG

1847.

Fig. 18. Graßmann's prize-winning treatise

Drobisch and Möbius, who were members of the Jablonowskian Society along with scientists such as W. Weber and G. Th. Fechner, were in charge of assessing the contributions to this scientific competition. Drobisch was quite skeptical of Graßmann's paper, in which Graßmann went beyond what he had written in *Extension Theory* and focused on the inner product of vectors: "Strictly speaking, it [Graßmann's paper – H.-J. P.] is not a reconstruction of the Leibnizian calculus ..."[162] Drobisch thought that the paper lacked practical applications of the theory which might have served to illustrate the usefulness of Graßmann's calculus. Möbius' opinion, in contrast, was more positive and thoughtful. Contradicting Drobisch, he judged: "It seems to me that there can be no doubt that the Leibnizian idea as we find it in this paper ... has recently been concretized by the mathematical operations ... invented by Graßmann in Stettin. At least, a first step has been taken, and who can predict where it will end?"[163] Furthermore, he emphasized that "the theory of what the author calls inner products proves that the author is a perspicacious man"[164].

Nevertheless, Möbius criticized two things. On the one hand, similar to Drobisch's objections, he thought that the paper lacked practical examples of how the theory could be applied; on the other hand, he criticized Graßmann's method. "Intuition is the foundation of geometry", he explained. Möbius pointed out that, in Graßmann's paper, "in analogy with arithmetic, some objects are dealt with as if they were magnitudes, even though – strictly speaking – they are not, and in some cases it is impossible to create a mental image of these objects"[165].

Möbius' criticism was not completely off the mark. Graßmann's thinking tended to be very abstract, and this sometimes led him to formal concepts which could not be interpreted geometrically and which Möbius therefore called "pseudo-magnitudes".[166] Despite these shortcomings, Graßmann's prize-winning treatise was much easier to understand than *Extension Theory*.

Thanks to the discerning judgment of Möbius, Graßmann received the prize of the Jablonowskian Society on the 200th birthday of Leibniz. Drobisch wrote a letter to the winner. This letter must have strengthened Graßmann's confidence in his mathematical ideas: "I hope that your findings", Drobisch told Graßmann, "will prove their worth in useful applications and that this will convince those readers ... who have been discouraged from studying your treatise by its abstract philosophical foundation."[167]

Although intensely focused on mathematics, Graßmann nevertheless found time for other projects. Apart from his everyday work at school, he and a colleague put together a collection of texts for young readers in 1846 (Graßmann/Langbein 1868). It

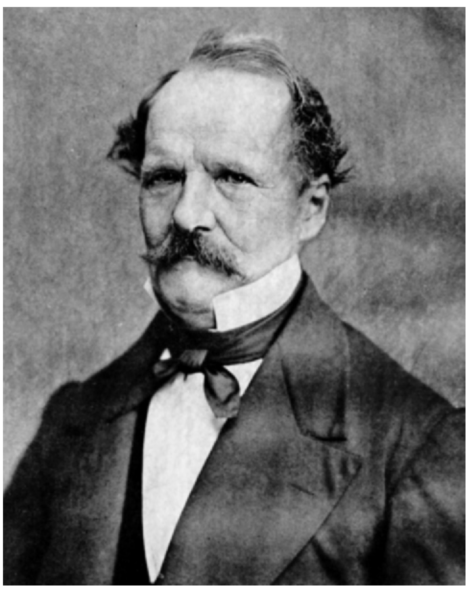


Fig. 19. Ernst Eduard Kummer (1810 – 1893)

was reprinted a number of times and very popular in Stettin. In May 1847, Graßmann's good work as a teacher was rewarded with a promotion in the school hierarchy.

In 1845, Graßmann spent time working on statutes for the Pomeranian church charter. He was also an active member of philanthropic and charitable organizations. He and his brother Robert developed statutes for these organizations as well.

Since his paper on the Leibnizian geometrical characteristic had been a success, the two brothers set out on a collaboration in which they intended to reconstruct mathematics.[168] Hermann Graßmann was now feeling the power of his gift for mathematics. His only wish, by now, was to spend his entire time on mathematics. Therefore, Graßmann made an unusually bold move. In May 1847, he sent a letter to the Minister of Culture and Education, Eichhorn, asking the minister to consider him a candidate should a position become available at a Prussian university. He gave the minister a brief overview of his career in mathematics, informing Eichhorn that, unfortunately, his work as a teacher took up too much of his time and that he was therefore unable to develop his new mathematical methods. Referring to his recent work, he wrote: "I also realized that this newly discovered branch of science made a fundamental reconstruction of mathematics necessary. This reconstruction is likely to change the entire field of applied mathematics, including the teaching of mathematics in schools. … It is not just that I have little hope that my time will permit me to continue even the smallest portion of my work. I also lack the possibility of communicating with interested colleagues, which I need to feel optimistic about my work. Only in such a state of mind could I hope to continue my research and find the perfect form for my theory."[169] Graßmann also pointed to his age, which was a pressing issue in his plans for a new career. He included *Extension Theory* and the prize-winning treatise, asking the minister "to show them to impartial mathematical specialists"[170]. On 17 May 1847, Eichhorn did in fact send the books to E. E. Kummer, who returned them on 12 June – less than a month later – along with a written assessment. Kummer's assessment of the books had catastrophic consequences for Graßmann's hopes of finding a position in academia.

With respect to *Extension Theory*, Kummer emphasized that Graßmann's vision of representing the position and size of spatial objects in "symbolic formulas" was "nothing new": "It has been carried out in many different ways in the past without anybody seeing the need for a new mathematical discipline"[171]. According to Kummer, Graßmann's project was "in itself neither a good nor a bad one, because the only thing that counts is how it is executed, and if we wanted to assess the scientific value of its execution, form and content would have to be the decisive criteria"[172].

In Kummer's opinion, the book was a failure from the formal point of view, but he admitted that he liked its style. He disliked the book's general structure and criticized Graßmann's "truly despicable habit being very original in inventing new words, which he then [uses] instead of the traditional terminology … even when there is no good reason to do so"[173]. And, unfortunately, this view would prevail for decades. According to Kummer, there could be no doubt that "mathematicians will continue to ignore this book. It takes enormous effort to become familiar with the book, and the insights it seems to offer are really not worth such an effort"[174]. Concerning the content of the book, Kummer noted that its form made it hard to judge but that he had encountered "new and interesting ideas"[175]. Kummer's overall assessment of Graßmann's potential as a university scholar was negative: Firstly, his works lacked clarity. Secondly, Graßmann's education and his field of study were too limited. Thirdly, there were other young mathematicians whose capabilities and achievements made them a better choice for such a position. Nevertheless, Kummer believed that Graßmann's achievements were sufficient to grant him the title of professor.

Eichhorn, of course, followed Kummer's advice. But Graßmann had to wait another five years before he received the title of professor. The school administration in Stettin notified the ministry in Berlin that "Graßmann's current position in the faculty makes it seem inappropriate to grant him such authority. Graßmann has the rank of a fifth teacher … and none of the higher-ranking colleagues, not even the headmaster, are professors"[176].

This episode could have been a decisive turning point in Graßmann's life. But Graßmann had not achieved anything. The minister of culture and education sent him a diplomatic and evasive reply, saying that an academic position was "currently not likely to be an option"[177] and that such an opportunity was unlikely to appear in the near future.

This meant that all of Graßmann's attempts to disseminate his mathematical ideas had failed.

1.6 The Revolution comes to Germany (1848)

The political conflicts and upheavals announcing the Revolution of 1848 rudely confronted Hermann Graßmann with political reality. When, on 21 April 1847, bakeries, butcher stores and a large indoor market in Berlin were stormed by the masses, Stettin was ripped from its small-town slumber. On 24 April, parts of the local population, enraged by unemployment and inflation, plundered bakeries before the eyes of the helpless police force. The military had to intervene. Shots were fired into the mass of protesters. Most of the city's bourgeois population, whose everyday business had ground to a halt, supported the military: "At night, armed citizens stood watch. The following day, there were more attempted lootings and violent responses. This 'potato revolution' brought a persistent feeling of unease over the economic life of Stettin. Shopowners had to be protected by the police or the military, fixed prices for potatoes had to be introduced,

Fig. 20. The hunger revolt in Stettin. Engraving by G. Nicholls 1847

exports were forbidden and taxes on grain mills temporarily abolished. The agitators and protesters were brought to justice."[178]

Hermann Graßmann had been immersed in reconstructing mathematics and obtaining a professorship when the political situation in Stettin suddenly got tense. Apart from discussions about the roots and possible consequences of the hunger revolts, the citizens of Stettin began to pay more attention to the city's economy.

Three problems were at the center of the heated public discussions in Stettin in 1846/47:

Firstly, there were questions related to political efforts which aimed to facilitate commerce among the German states. This issue was extremely important for businesses in Stettin.

Secondly, the railway connection to Berlin was an object of debate. It had been completed in 1844, and its repercussions on the city's economy were still unclear. The citizens of Stettin feared the competition from Berlin and wondered what effect it might have on local trade and industry.

Thirdly, merchants in Stettin disagreed about protectionist customs regulations and the issue of free trade. This dilemma split the community of merchants into two factions.[179]

During the crisis of 1847/48, interest in economic policies grew, while political demands concerning the nation as a whole were still rare. But, nevertheless, voices in favor of a Prussian constitution, which Frederick William III had promised and Frederick William IV had violently rejected, were getting louder. Carefully and reluctantly, the citizens of Stettin closed their ranks and, in 1847, a democratic journal called "Wächter der Ostsee" ("Baltic Guardian") did what it could to spread its message, always hindered by censorship.

Up to this point, Hermann Graßmann had devoted himself entirely to school and science, religion and charitable organizations. The "so-called issues of everyday politics"[180] had left him cold. But now he began to concern himself with what was at stake politically. Together with his brother, he studied Dahlmann's *Politics* ("Politik", 1835) and Schleiermacher's *Theory of the State* ("Die Lehre vom Staat", 1845).

Both books had been written by liberal bourgeois intellectuals and aimed to find a balance between bourgeoisie and aristocracy based on constitutional monarchy. Schleiermacher and Dahlmann were in favor of a political development that would strengthen the political position of the bourgeoisie within the monarchy. Certain elements from the Romantic philosophy of nature gave their political theories a philosophical foundation.[181] These elements served to explain why it was necessary to *unite* the contrary forces of bourgeoisie and aristocracy. The two works brought forward both anti-despotic and anti-revolutionary positions, reminding Germans of their "historical development" and "political organism" (implicitly taking Great Britain as a model). National government was to reflect these historical and political factors and represent the interests of the bourgeoisie.

In the mid-1830s, these views, which mostly had arisen prior to 1835, could still be called progressive in the German context. But they lost more and more of their progressiveness when a truly revolutionary historical situation began to emerge from class conflicts and changes in the distribution of political power. Since the liberal bourgeoi-

In his *Politics* (1835), Dahlmann wrote: "...revolutions are not just symptoms of grave misfortune which has come over the state, and not just a unilateral failure, but revolutions as such are a misfortune and a failure. Therefore, wise and responsible men, because it guarantees them freedom from prosecution, will never justify their deeds by declaring the revolution a success. Also, they will never engage in criminal activities just because it is the only way to escape humiliation. For what initially was directed only against the sovereign or the dynasty can easily overthrow the entire social order, and even if the new rulers really do try to show better will, can these rulers really remain in power? But once a revolution has become inevitable, even the patriot who initially opposed it does the right thing to join it because a situation in which there is no government, because everybody represents the government, is intolerable..."[182]

Fig. 21. Friedrich Christoph Dahlmann (1785 – 1860)

Die Politik,

auf den Grund und das Maß der gegebenen Zustände zurückgeführt.

Von

F. C. Dahlmann.

Erster Band.

Zweite verbesserte Auflage.

Leipzig,
Weidmann'sche Buchhandlung.
1847.

Fig. 22. Title page of Dahlmann's *Politics*

sie – of which Dahlmann was a member – remained loyal to these old convictions, it violently opposed the Revolution of 1848. Also, in terms of social class, the German bourgeoisie was in a completely different situation than the French during the French Revolution: "Unlike the French bourgeoisie of 1789, the Prussian bourgeoisie, when it confronted monarchy and aristocracy, the representatives of the old society, was not a class speaking for the whole of modern society. It had been reduced to a kind of estate as clearly distinct from the Crown as it was from the people... From the first it was inclined to betray the people and to compromise with the crowned representatives of the old society, for it already belonged itself to the old society; it did not advance the interests of a new society against an old one, but represented refurbished interests within an obsolete society. [It was] revolutionary with regard to conservatives and conservative with regard to the revolutionaries."[183]

The political views and positions which Hermann Graßmann found in Dahlmann and Schleiermacher did nothing to increase his solidarity with revolutionary and democratic politics. In fact, he was not even completely convinced of Schleiermacher's and Dahlmann's bourgeois liberalism.

This mainly had to do with the social and economic underdevelopment of the town of Stettin and with Graßmann's social position as an employee of the Prussian schoolsystem.[184] Despite the fact that in 1847/48 Berlin was economically far more advanced than Stettin, Marx and Engels nevertheless chose a different location to publish their revolutionary and democratic "Neue Rheinische Zeitung": "The Berlin of that time we knew only too well from our own observation, with its hardly hatched bourgeoisie, its cringing petty bourgeoisie, audacious in words but craven in deeds, its still wholly underdeveloped workers, its mass of bureaucrats, aristocratic and court riff-raff ..."[185] This description of Berlin just goes to show that revolutionary and democratic sentiment was unlikely to take hold in Stettin, which was much smaller.

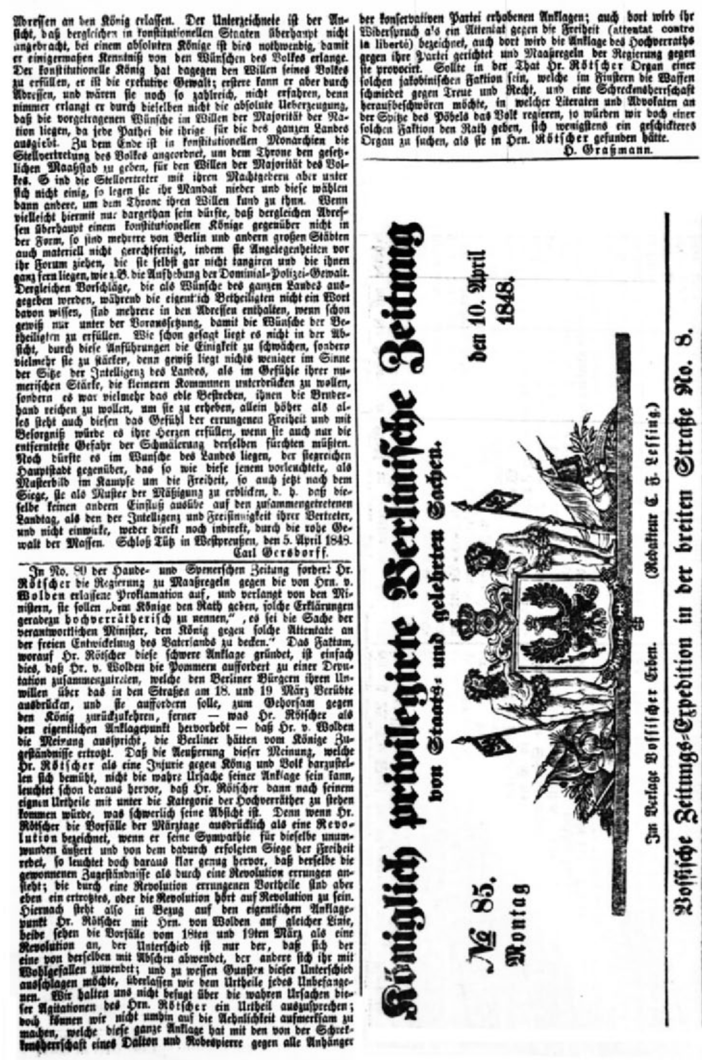
When, in 1848, revolutionary uprisings shook all of Germany, Stettin remained calm. Debates in social clubs, political discussions in the newspapers and smaller strikes, mostly among young carpenters in shipbuilding[186], were the only exceptions.

There is no need to repeat what has been said about the social and political situation in Stettin. Nevertheless, we should remember that most secondary-school professors were loyal to the constitution and monarchy.[187] Almost all of Hermann Graßmann's colleagues and friends were, as far as we know, conservatives.

The friendly ties between the Graßmann and Droysen families, which had existed for decades, were a typical example. When Hermann Graßmann's mother and her children fled to her mother's house during the Liberation War of 1813/14, J. G. Droysen's father was the local minister. J. G. Droysen was a leading historian and in favor of the "lesser German solution", that is, of unifying Germany without Austria. He opposed the revolutionary tendencies of his time and became one of the ideological fathers of Bismarck's political agenda. His autobiography tells us many things about the two families: "They had gotten to know each other in Greifenhagen, and ever since the Graßmann and Droysen families had been on friendly terms. All his life, the minister's son was thankful to this honorable and loyal man [J. G. Graßmann – H.-J. P.], who had been his most important teacher ..."[188]

Carl Gottfried Scheibert, Graßmann's brother-in-law and headmaster of the "Friedrich-Wilhelmsschule" in Stettin, is another important name. He was an extremely conservative person and despised all liberal and democratic tendencies (see Schulze 1939). In 1848/49, he became a conservative representative in the parliament of Erfurt. In 1847, in a letter to Suffrian, he raged against the times as being "addicted to constitutionalism and drunk with liberalism ..."[189]. In 1855, he became an administrator for Protestant secondary-schools in the province of Silesia, but gave up this position for political reasons in 1873.

Hermann Graßmann's political views were firmly linked to the socio-economic situation of Prussia, and especially of Stettin. Graßmann, who brought a revolution to mathematics, did not even come close to understanding the true importance of the revolution which shook Germany in March 1848. When, in February 1848, Paris was hit by a revolution so violent that it "upset ... the very same sort of Government which the Prussian bourgeoisie was going to set up in their own country"[190], Hermann Graßmann became convinced once and for all that revolutions were not a political option. In 1853, he wrote a letter to Möbius apologizing for being out of touch and explaining that



Adressen an den König erlassen. Der Unterzeichnete ist der Ansicht, daß dergleichen in konstitutionellen Staaten überhaupt nicht angebracht, bei einem absoluten Könige ist dies nothwendig, damit er einigermaßen Kenntniß von den Wünschen des Volkes erlange. Der konstitutionelle König hat dagegen den Willen seines Volkes zu erfüllen, er ist die exekutive Gewalt; erstere kann er aber durch Adressen, und wären sie noch so zahlreich, nicht erfahren, denn nimmer erlangt er durch dieselben nicht die absolute Ueberzeugung, daß die vorgetragenen Wünsche im Willen der Majorität der Nation liegen, da jede Pathei die ihrige für die des ganzen Landes ausgiebt. Zu dem Ende ist in konstitutionellen Monarchien die Stellvertretung des Volkes angeordnet, um dem Throne den gesetzlichen Maaßstab zu geben, für den Willen der Majorität des Volkes. Sind die Stellvertreter mit ihren Machtgebern aber unter sich nicht einig, so legen sie ihr Mandat nieder und diese wählen dann andere, um dem Throne ihren Willen kund zu thun. Wenn vielleicht hiermit nur dargethan sein dürfte, daß dergleichen Adressen überhaupt einem konstitutionellen Könige gegenüber nicht in der Form, so sind mehrere von Berlin und andern großen Städten auch materiell nicht gerechtfertigt, indem sie Angelegenheiten vor ihr Forum ziehen, die sie selbst gar nicht tangiren und die ihnen ganz fern liegen, wie z. B. die Aufhebung der Dominial-Polizei-Gewalt. Dergleichen Vorschläge, die als Wünsche des ganzen Landes ausgegeben werden, während die eigentlich Betheiligten nicht ein Wort davon wissen, sind mehrere in den Adressen enthalten, wenn schon gewiß nur unter der Voraussetzung, damit die Wünsche der Betheiligten zu erfüllen. Wie schon gesagt liegt es nicht in der Absicht, durch diese Anführungen die Einigkeit zu schwächen, sondern vielmehr sie zu stärken, denn gewiß liegt nichts weniger im Sinne der Sitze der Intelligenz des Landes, als im Gefühle ihrer numerischen Stärke, die kleineren Kommunen unterdrücken zu wollen, sondern es war vielmehr das edle Bestreben, ihnen die Bruderhand reichen zu wollen, um sie zu erheben, allein höher als alles steht auch diesen das Gefühl der errungenen Freiheit und mit Besorgniß würde es ihre Herzen erfüllen, wenn sie auch nur die entfernteste Gefahr der Schmälerung derselben fürchten müßten. Noch dürfte es im Wunsche des Landes liegen, der siegreichen Hauptstadt gegenüber, das so wie diese jenem vorleuchtete, als Musterbild im Kampfe um die Freiheit, so auch jetzt nach dem Siege, sie als Muster der Mäßigung zu erblicken, d. h. daß dieselbe keinen andern Einfluß ausübe auf den zusammengetretenen Landtag, als den der Intelligenz und Freisinnigkeit ihrer Vertreter, und nicht einwirke, weder direkt noch indirekt, durch die rohe Gewalt der Massen. Schloß Tütz in Westpreußen, den 5. April 1848.

Carl Gersdorff.

In No. 80 der Haude- und Spenerschen Zeitung fordert Hr. Rötscher die Regierung zu Maaßregeln gegen die von Hrn. v. Wolden erlassene Proklamation auf, und verlangt von den Ministern, sie sollen „dem Könige den Rath geben, solche Erklärungen geradezu hochverrätherisch zu nennen," „es sei die Sache der verantwortlichen Minister, den König gegen solche Attentate an der freien Entwickelung des Vaterlands zu decken." Das Faktum, worauf Hr. Rötscher diese schwere Anklage gründet, ist einfach dies, daß Hr. v. Wolden die Pommern auffordert zu einer Deputation zusammenzutreten, welche den Berliner Bürgern ihren Unwillen über das in den Straßen am 18. und 19 März Verübte ausdrücken, und sie auffordern solle, zum Gehorsam gegen den König zurückzukehren, ferner — was Hr. Rötscher als den eigentlichen Anklagepunkt hervorhebt — daß Hr. v. Wolden die Meinung ausspricht, die Berliner hätten vom Könige Zugeständnisse ertrotzt. Daß die Aeußerung dieser Meinung, welche Hr. Rötscher als eine Injurie gegen König und Volk darzustellen sich bemüht, nicht die wahre Ursache seiner Anklage sein kann, leuchtet schon daraus hervor, daß Hr. Rötscher dann nach seinem eignen Urtheile mit unter die Kategorie der Hochverräther zu stehen kommen würde, was schwerlich seine Absicht ist. Denn wenn Hr. Rötscher die Vorfälle der Märztage ausdrücklich als eine Revolution bezeichnet, wenn er seine Sympathie für dieselbe unumwunden äußert und von dem dadurch erfolgten Siege der Freiheit redet, so leuchtet doch daraus klar genug hervor, daß derselbe die gewonnenen Zugeständnisse als durch eine Revolution errungen ansieht; die durch eine Revolution errungenen Vortheile sind aber eben ein ertrotztes, oder die Revolution hört auf Revolution zu sein. Hiernach steht also in Bezug auf den eigentlichen Anklagepunkt Hr. Rötscher mit Hrn. von Wolden auf gleicher Linie, beide sehen die Vorfälle vom 18ten und 19ten März als eine Revolution an, der Unterschied ist nur der, daß sich der eine von derselben mit Abscheu abwendet, der andere sich ihr mit Wohlgefallen zuwendet; und zu wessen Gunsten dieser Unterschied ausschlagen möchte, überlassen wir dem Urtheile jedes Unbefangenen. Wir halten uns nicht befugt über die wahren Ursachen dieser Agitationen des Hrn. Rötscher ein Urtheil auszusprechen; doch können wir nicht umhin auf die Aehnlichkeit aufmerksam zu machen, welche diese ganze Anklage hat mit den von der Schreckensherrschaft eines Dalton und Robespierre gegen alle Anhänger der konservativen Partei erhobenen Anklagen; auch dort wird ihr Widerspruch als ein Attentat gegen die Freiheit (attentat contre la liberté) bezeichnet, auch dort wird die Anklage des Hochverraths gegen ihre Partei gerichtet und Maaßregeln der Regierung gegen sie provocirt. Sollte in der That Hr. Rötscher Organ einer solchen jakobinischen Faktion sein, welche im Finstern die Waffen schmiedet gegen Treue und Recht, und eine Schreckensherrschaft heraufbeschwören möchte, in welcher Literaten und Advokaten an der Spitze des Pöbels das Volk regieren, so würden wir doch einer solchen Faktion den Rath geben, sich wenigstens ein geschickteres Organ zu suchen, als sie in Hrn. Rötscher gefunden hätte.

H. Graßmann.

Königlich privilegirte Berlinische Zeitung

von Staats- und gelehrten Sachen.

№ 85. Montag

den 10. April 1848.

Im Verlage Vossischer Erben. (Redakteur C. G. Lessing)

Vossische Zeitungs-Expedition in der breiten Straße No. 8.

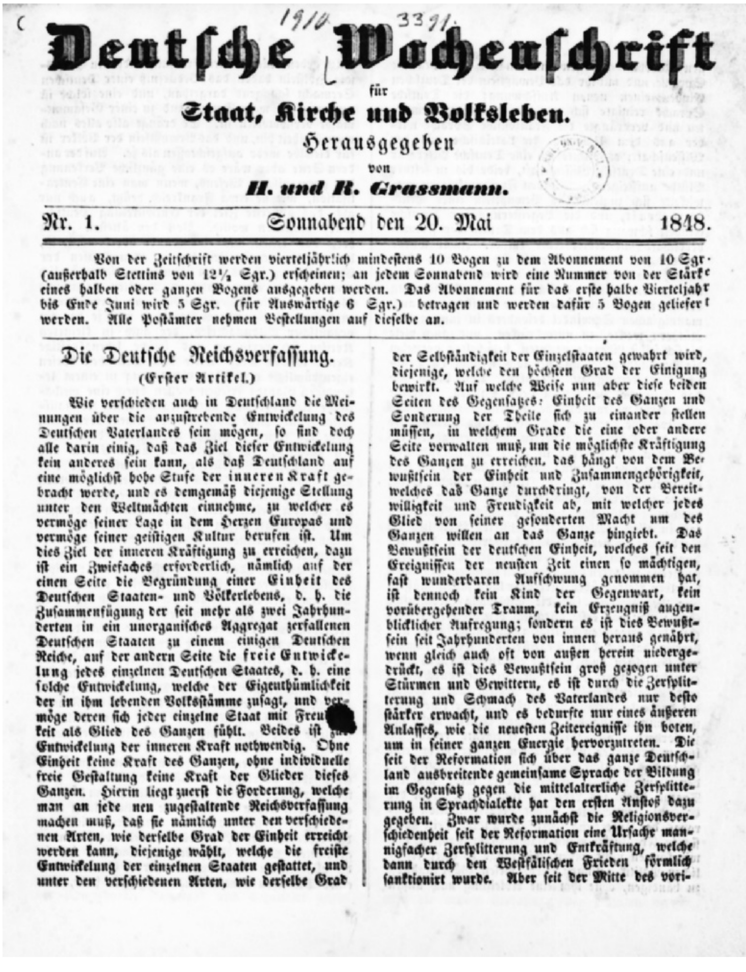
Fig. 23. A letter to the editor by Hermann Graßmann in the "Vossische Zeitung"

"for an extended period of time, the events of 1848 distracted me from my mathematical work, drawing my attention to other issues". And Graßmann went on to say: "Up to that date, I had kept my distance from the news of the day, and so-called everyday politics had been of no interest to me. But the revolutionary movements of that year, which I despised deeply, seemed to oblige me to help the small group of people who were fighting this uprising. I also believed that I had to help create a healthier, more vigorous bureaucratic system to replace the system which, in my view, was mainly responsible for this outbreak of democratic fanaticism."[191]

On 15 April 1848, Graßmann published an article called *The Fruits of the Barricades in Berlin* ("Die Früchte des Berliner Barrikadenkampfes") in a Stettin newspaper. In this article, Graßmann's anti-revolutionary views and his blindness to the political situation were even more extreme. Writing about a parade honoring the fallen revolutionists, which took place on 19 March, Graßmann got carried away by his monarchist convictions: "… the raging mob dragged the bodies of the fallen to the palace, uncovered their wounds and, as if all human empathy had left their hearts, aggressively demanded to see the king and screamed for revenge. And the citizens of Berlin? They let it all happen and did nothing to stop these atrocities. This is what the king got for his trust and unconditional love. A word from the king, and Berlin would have lain in ruins. … But would Berlin awake from its trance? Would it call these agitators by their name, would it call them traitors? No, the presumptuous city was and is completely unwilling to do so. Instead, it will try to glorify the agitators and build a monument to commemorate them for ever. I would like to propose an inscription: To the fallen traitors, in remembrance of our shared disgrace, the citizens of Berlin. … I hope that at last all clear-headed people will raise their voices and openly condemn the deeds of those who use the holy concept

Robert Graßmann proudly wrote in his curriculum vitae:

"On a number of occasions during this period of time, the author made positive contributions to political life. When, in 1848, the National Assembly strayed on a dangerous path and, in November of 1848, the majority of its members remained in session despite the fact that it had been dissolved, the author rushed to Berlin on 11 November. … He had a long talk with a number of members of the center-right block and learned that the majority of the members of the National Assembly were leaning towards the republic. … Therefore, on the following morning, he rushed to see the former president of the National Assembly, Grabow, and the minister of the interior, von Manteuffel. He also traveled to Potsdam to see members of the Court and Deputy Bassermann from Mannheim, … learning more about the state of the nation. These men agreed that the National Assembly had to be stopped from what it was up to. Mr. von Manteuffel was in favor of adopting the Belgian constitution. The author pointed out to von Manteuffel that it made much more sense to propose the constitution which the National Assembly had voted on as a temporary basic law for the state before gradually improving it. Von Manteuffel accepted this argument and acted accordingly. While the author was absent, the administrative bodies in Stettin, terrorized by the so-called Party of Progress, had come out against the government and in favor of the National Assembly. … Now it had become the author's duty to lead the citizens of Stettin back on the right path. The author and his friends successfully did so. The city council of Stettin annulled the bill of 16 November and declared itself … loyal to the government and against the National Assembly. … The danger was over."[193]



Deutsche Wochenschrift
für
Staat, Kirche und Volksleben.
Herausgegeben
von
H. und R. Grassmann.

Nr. 1. Sonnabend den 20. Mai 1848.

Von der Zeitschrift werden vierteljährlich mindestens 10 Bogen zu dem Abonnement von 10 Sgr. (außerhalb Stettins von 12½ Sgr.) erscheinen; an jedem Sonnabend wird eine Nummer von der Stärke eines halben oder ganzen Bogens ausgegeben werden. Das Abonnement für das erste halbe Vierteljahr bis Ende Juni wird 5 Sgr. (für Auswärtige 6 Sgr.) betragen und werden dafür 5 Bogen geliefert werden. Alle Postämter nehmen Bestellungen auf dieselbe an.

Die Deutsche Reichsverfassung.
(Erster Artikel.)

Wie verschieden auch in Deutschland die Meinungen über die anzustrebende Entwickelung des Deutschen Vaterlandes sein mögen, so sind doch alle darin einig, daß das Ziel dieser Entwickelung kein anderes sein kann, als daß Deutschland auf eine möglichst hohe Stufe der inneren Kraft gebracht werde, und es demgemäß diejenige Stellung unter den Weltmächten einnehme, zu welcher es vermöge seiner Lage in dem Herzen Europas und vermöge seiner geistigen Kultur berufen ist. Um dies Ziel der inneren Kräftigung zu erreichen, dazu ist ein Zwiefaches erforderlich, nämlich auf der einen Seite die Begründung einer Einheit des Deutschen Staaten- und Völkerlebens, d. h. die Zusammenfügung der seit mehr als zwei Jahrhunderten in ein unorganisches Aggregat zerfallenen Deutschen Staaten zu einem einigen Deutschen Reiche, auf der andern Seite die freie Entwickelung jedes einzelnen Deutschen Staates, d. h. eine solche Entwickelung, welche der Eigenthümlichkeit der in ihm lebenden Volksstämme zusagt, und vermöge deren sich jeder einzelne Staat mit Freu[illegible]keit als Glied des Ganzen fühlt. Beides ist [illegible] Entwickelung der inneren Kraft nothwendig. Ohne Einheit keine Kraft des Ganzen, ohne individuelle freie Gestaltung keine Kraft der Glieder dieses Ganzen. Hierin liegt zuerst die Forderung, welche man an jede neu zugestaltende Reichsverfassung machen muß, daß sie nämlich unter den verschiedenen Arten, wie derselbe Grad der Einheit erreicht werden kann, diejenige wählt, welche die freiste Entwickelung der einzelnen Staaten gestattet, und unter den verschiedenen Arten, wie derselbe Grad der Selbständigkeit der Einzelstaaten gewahrt wird, diejenige, welche den höchsten Grad der Einigung bewirkt. Auf welche Weise nun aber diese beiden Seiten des Gegensatzes: Einheit des Ganzen und Sonderung der Theile sich zu einander stellen müssen, in welchem Grade die eine oder andere Seite vorwalten muß, um die möglichste Kräftigung des Ganzen zu erreichen, das hängt von dem Bewußtsein der Einheit und Zusammengehörigkeit, welches das Ganze durchdringt, von der Bereitwilligkeit und Freudigkeit ab, mit welcher jedes Glied von seiner gesonderten Macht um des Ganzen willen an das Ganze hingiebt. Das Bewußtsein der deutschen Einheit, welches seit den Ereignissen der neusten Zeit einen so mächtigen, fast wunderbaren Aufschwung genommen hat, ist dennoch kein Kind der Gegenwart, kein vorübergehender Traum, kein Erzeugniß augenblicklicher Aufregung; sondern es ist dies Bewußtsein seit Jahrhunderten von innen heraus genährt, wenn gleich auch oft von außen herein niedergedrückt, es ist dies Bewußtsein groß gezogen unter Stürmen und Gewittern, es ist durch die Zersplitterung und Schmach des Vaterlandes nur desto stärker erwacht, und es bedurfte nur eines äußeren Anlasses, wie die neuesten Zeitereignisse ihn boten, um in seiner ganzen Energie hervorzutreten. Die seit der Reformation sich über das ganze Deutschland ausbreitende gemeinsame Sprache der Bildung im Gegensatz gegen die mittelalterliche Zersplitterung in Sprachdialekte hat den ersten Anstoß dazu gegeben. Zwar wurde zunächst die Religionsverschiedenheit seit der Reformation eine Ursache mannigfacher Zersplitterung und Entkräftung, welche dann durch den Westfälischen Frieden förmlich sanktionirt wurde. Aber seit der Mitte des vori-

Fig. 24. Title page of the first issue of the political weekly journal edited by Hermann and Robert Graßmann

of Human Rights to justify unrest and aggression. I hope that the most intense feeling of disdain for all revolutionary activities will penetrate all social classes. Not before this has occurred can there be any hope that Prussia will rise above this internal upheaval and return to the secure path of free patriotic development."[192]

Graßmann was very sincere when he declared himself an enemy of the Revolution and completely committed to his convictions. In May 1848, Hermann and Robert Graßmann founded the "German Weekly for Politics, Religion and National Life" ("Deutsche Wochenschrift für Staat, Kirche und Volksleben", 1848). Hermann Graßmann spent a large portion of his assets on the project – more than 2000 taler. The journal gave expression to the brothers' political ideas: constitutional monarchy, the "free development of the nation" and an anti-revolutionary, anti-democratic position somewhere between liberalism and conservatism. The journal's motto was to fight for "true liberty, based on morality and respect for the law, to fight for the dynamic forces of liberty and justice … and resist revolution and reaction …"[194]

The Graßmann brothers developed their political ideas together but – unfortunately – did not sign their articles in the "Deutsche Wochenschrift". What did their political concepts look like?

According to the Graßmanns, the wars of liberation against the Napoleonic occupiers had opened a new chapter in German history. This conflict had forced the monarchy to take the middle class more seriously. Reformers had established greater liberties for the cities and a national legislation which permitted popular armed resistance.[195] The period of feudalistic rollback which began in 1820 had been brought about by "incompetent princes and government officials"[196]. "A time of distrust had come over Germany, a time in which every free expression of popular sentiment provoked suspicions on the part of the fearful rulers. The young tree of free popular institutions which had just been planted was cut down from the top and robbed of its crown. Only when the present king [Frederick William IV – H.-J. P.] came to power did the work on a national organism continue."[197] The trust in a "Romantic king" was unbroken in 1848. It was accompanied by enthusiasm for "authentic" German values: "… while those who called themselves liberals smirked arrogantly when they heard talk of German values and were incapable of understanding what German enthusiasm meant."[198] The philosophy of nature provided the concepts which served to reject the political developments in France, especially the Revolution of February 1848. Germany was compared to a plant and France to a crystal or an inanimate mechanism.[199]

According to Graßmann, "it would be completely contrary to Germanic nature should centralization, as we find it in France, be proposed as a goal for Germany, even if it were considered the least important goal."[200] True to his anti-revolutionary attitude, he said elsewhere: "… if we gave up on organic structures and aimed to replace them with mechanisms from foreign nations, this would be a brutish form of vandalism making the destruction of Classical art treasures by hordes of barbarians seem insignificant."[201]

What was believed to be the rediscovery of "Germanic nature" led the Graßmanns to extreme nationalism: "… Germany must refrain from blindly imitating foreign nations. Germany must understand the mission it has been given, namely to remain true to its form of development, unique among the nations and adequate to the highest organic spirit. In the history of nations, it will hereby create a new and unique model for the world."[202] After a few additional theoretical turns, the Germans appeared as the "chosen people" and Prussia as the leading German nation: "In short, if we take Prussia as the leading German nation, Prussia will become Germany, and Germany, Prussia. The German Reich will be the heart and the hub of Europe."[203]

Pre-Hegelian dialectics served as a foundation for this Romantic interpretation of the social reform movement: "If we want its [Prussia's – H.-J. P.] Constitution to be strong, its legislative structures must contain the double preconditions of all life, namely the will to *maintain* its unique form of existence, its unity as a form of life, and the will to *develop* new and fresh forms of life. These two elements, one of which is conservative, the other progressive and given to reforms, therefore represent the essential preconditions for vigorous legislation."[204]

This view of society united Romanticism and biological thinking, which led the Graßmanns to adopt a pro-aristocratic position: "… the prince represents … the conservative element, the organic unity; the parliament, the progressive element, the development of life."[205] From a political point of view, the dialectical schema suggested a constitutional monarchy with a two-chamber parliament.

The Graßmanns' opinions on the ideal electoral system for Germany (and Prussia) were a consequence of their political position, which strove for a balance of power between aristocracy and bourgeoisie and sought to keep the mass of the population away from power. Relying on liberal bourgeois political theory, the Graßmanns attempted to find a harmonious interpretation for all possible conflicts – whether they arose from a clash of feudal and bourgeois interests or from a closing of the ranks against the working class. They emphasized that "it is a typical sign of healthy political development when oppositions arise on all sides. But, depending on what kind of object these oppositions refer to, they must constantly renew and annul themselves, merge in various ways and separate from each other, as in a living being. But when all these oppositions are annulled by *one* single opposition, … then political life is frozen and has been ripped apart, the organism of the state has become a skeleton and the thriving life of a free nation has been reduced to pathetic, disunited factions."[206]

The Graßmanns rejected political parties and partisanship because they believed in "natural and fundamentally indelible differences of social rank which only communism would attempt to wipe out …"[207] In their petit-bourgeois intellectual aloofness, they felt that a universal and direct right to vote threatened to "annihilate all culture" and bring a "tyrannical mob" to power. All "higher interests", the Graßmanns wrote, appeared in a "small number of people": "In this type of political representation, all higher values would be increasingly repressed by the workingman's material interests. One would not get a democracy but the rule of a tyrannical mob, an ochlocracy. It is a complete mystery to me how anybody could view this return to mob rule as progress or consider this restriction and repression of all higher values by the predominating interests of the workers the best foundation for a free constitution."[208] This was an expression of contempt for the most important aims of the Revolution of 1848 and the Graßmanns thus denied the people the right to harvest the fruits of their bloody struggle.

The Graßmanns also considered ballot voting too mechanical and too impersonal a technique. They opted for a hierarchical system with a permanent electoral college. Concerning the actual electoral system – on the occasion of elections for the lower house of the Prussian parliament – the brothers sympathized with the bourgeoisie. They favored the liberal bourgeoisie, stating that "the value of all productive activities, as it appears to the entire nation, and the political weight the nation grants the producers, are directly linked to the producer's *income* ..."[209] Income tax was to justify the increased political power of the bourgeois middle class: "The weight of every vote in parliamentary elections should correspond directly to the taxes the individual has to pay, after the minimum of money the individual needs to survive has been added to his tax load. Everybody may feel free to pay a greater amount of taxes than he must."[210] Graßmann did not forget to tell the ruling class that this electoral system posed no threat to its political power. Since the working class was responsible for 45 % of the nation's tax revenue, Graßmann believed that they were at "no disadvantage"[211]. But the workers would also never be able to dominate the political process and the middle class kept the upper hand. The Graßmanns believed that local and regional interests were highly relevant for the overall political picture. This was yet another expression of their petit-bourgeois attitude. According to Engels[212], provincial life was where the petite bourgeoisie was at home. Political visions from the Middle Ages were part of this small-town mindset. They included the administrative independence of towns and communities ("The law should permit all citizens to participate in the local administration ..."[213]) and, as in ancient Germanic traditions, the idea of a council of adult men which would meet in the villages and neighborhoods to discuss local issues such as poverty, self-defense, fire brigades, law enforcement, schooling, road maintenance, etc.

Despite the fact that Hermann Graßmann was a sincere and well-meaning person, he nevertheless was an enemy of political progress and supported the counter-revolutionary rollback. He opposed the "bureaucratic system" but accepted the economic reality of his time. He believed in a monarchist "popular" political development but rejected a democratic and bourgeois state. This is to say that he was completely ignorant of what was truly at stake in Germany.

At the time, Pomerania was economically underdeveloped and class identities were yet to arise. There were no classes or social estates to speak of, "but at most ... former estates and classes not yet born"[214]. The dominance of Romantic political theory was still unbroken, and the 1840s had seen its rise to the status of official ideology in the state of Prussia. Pre-Hegelian dialectics served as a useful tool in the natural sciences but offered no explanation for historical developments in society, such as the Revolution of 1848. These were the historical boundaries which Hermann Graßmann could not cross.

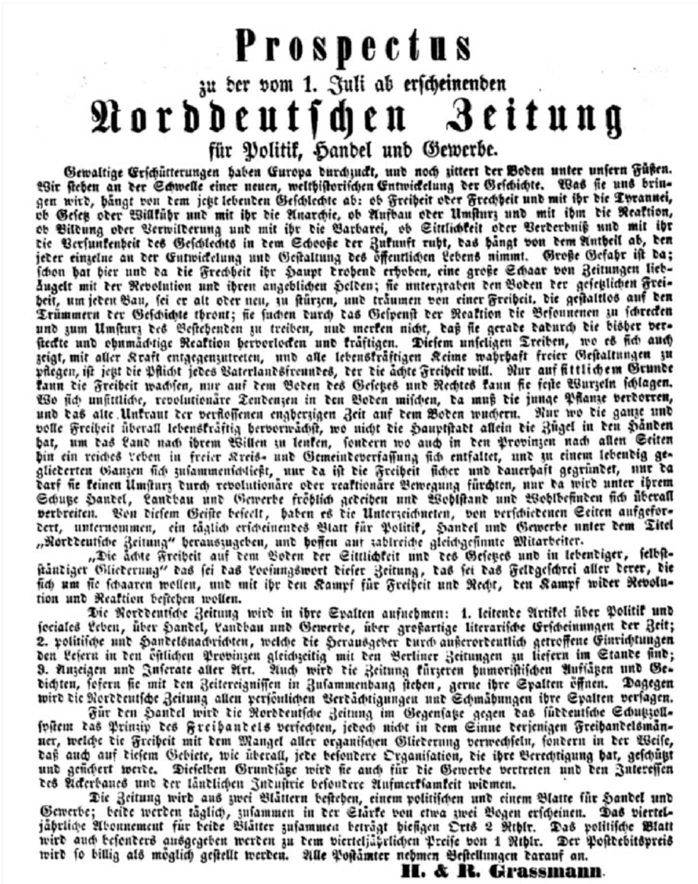
In 1834 **Heinrich Heine** explained the relationship between the Romantic philosophy of nature and political philosophy in his *History of Religion and Philosophy in Germany*:

"At the same time as Oken, the most genial thinker, and one of the greatest citizens of Germany, discovered new worlds of ideas, and inspired German youth about the original rights of humanity, about freedom and equality; alas! at the same time Adam Müller lectured on the stabling of humanity according to the principles of *Naturphilosophie*. At the same time, Mr. Görres preached the obscurantism of the Middle Ages, according to the scientific insight that the state is just a tree, and, in its organic structure, it must have a root, branches and leaves, all so nicely found in the corporative hierarchy of the Middle Ages; at the same time Mr. Steffens proclaimed the philosophical law according to which the peasantry was distinguished from the nobility because the peasant was determined by nature to work without enjoyment, the nobleman was entitled to enjoy without working..."[215]
"Alas, *Naturphilosophie*, which has brought forth the most magnificent fruit in certain areas of knowledge, especially in the true natural sciences, has, in other areas, produced the most pernicious weeds."[216]

Even though his worldview formed a relatively coherent unity, it was exposed to contradictory historical circumstances and therefore gave rise to two very contrary tendencies. One of these was a progressive approach to the natural sciences and mathematics, which relied on Schleiermacher's continuation of pre-Hegelian dialectics. But the other, linked to Graßmann's social identity, was his unprogressive political position, for which dialectics provided a philosophical foundation as well.[217] Graßmann's worldview mirrored the ambiguous nature of German life at the time, a situation in which progressive and reactionary elements came together. This was the Germany of which Marx said: "[I]t is every section of civil society which goes through a defeat before it celebrates victory and develops its own limitations before it overcomes the limitations facing it, asserts its narrow-hearted essence before it has been able to assert its magnanimous essence; thus the very opportunity of a great role has passed away before it is to hand ..."[218]

After the counter-revolutionary forces had succeeded and the king of Prussia had rejected the imperial crown, Hermann Graßmann began to feel frustrated with politics. The "popular" political developments he had hoped for had not materialized. His "unpractical" theoretical approach to the events of 1848/49 and the political theory accompanying this approach had turned out to be unrealistic.

The Graßmann brothers published six issues of their "German Weekly of Politics, Religion and National Life" between 20 May and 24 June 1848. It was replaced by a daily newspaper also edited by the two brothers, the "Norddeutsche Zeitung" ("North-German Newspaper"), the publication of which began on 1 July 1848. But Hermann began to wear down under the additional strain of working for the newspaper. On



Prospectus

zu der vom 1. Juli ab erscheinenden

Norddeutschen Zeitung

für Politik, Handel und Gewerbe.

Gewaltige Erschütterungen haben Europa durchzuckt, und noch zittert der Boden unter unsern Füßen. Wir stehen an der Schwelle einer neuen, welthistorischen Entwickelung der Geschichte. Was sie uns bringen wird, hängt von dem jetzt lebenden Geschlechte ab: ob Freiheit oder Frechheit und mit ihr die Tyrannei, ob Gesetz oder Willkühr und mit ihr die Anarchie, ob Aufbau oder Umsturz und mit ihm die Reaktion, ob Bildung oder Verwilderung und mit ihr die Barbarei, ob Sittlichkeit oder Verderbniß und mit ihr die Versunkenheit des Geschlechts in dem Schooße der Zukunft ruht, das hängt von dem Antheil ab, den jeder einzelne an der Entwickelung und Gestaltung des öffentlichen Lebens nimmt. Große Gefahr ist da; schon hat hier und da die Frechheit ihr Haupt drohend erhoben, eine große Schaar von Zeitungen liebäugelt mit der Revolution und ihren angeblichen Helden; sie untergraben den Boden der gesetzlichen Freiheit, um jeden Bau, sei er alt oder neu, zu stürzen, und träumen von einer Freiheit, die gestaltlos auf den Trümmern der Geschichte thront; sie suchen durch das Gespenst der Reaktion die Besonnenen zu schrecken und zum Umsturz des Bestehenden zu treiben, und merken nicht, daß sie gerade dadurch die bisher versteckte und ohnmächtige Reaktion hervorlocken und kräftigen. Diesem unseligen Treiben, wo es sich auch zeigt, mit aller Kraft entgegenzutreten, und alle lebenskräftigen Keime wahrhaft freier Gestaltungen zu pflegen, ist jetzt die Pflicht jedes Vaterlandsfreundes, der die ächte Freiheit will. Nur auf **sittlichem** Grunde kann die Freiheit wachsen, nur auf dem Boden des Gesetzes und Rechtes kann sie feste Wurzeln schlagen. Wo sich unsittliche, revolutionäre Tendenzen in den Boden mischen, da muß die junge Pflanze verdorren, und das alte Unkraut der verflossenen engherzigen Zeit auf dem Boden wuchern. Nur wo die ganze und volle Freiheit überall lebenskräftig hervorwächst, wo nicht die Hauptstadt allein die Zügel in den Händen hat, um das Land nach ihrem Willen zu lenken, sondern wo auch in den Provinzen nach allen Seiten hin ein reiches Leben in freier Kreis- und Gemeindeverfassung sich entfaltet, und zu einem lebendig gegliederten Ganzen sich zusammenschließt, nur da ist die Freiheit sicher und dauerhaft gegründet, nur da darf sie keinen Umsturz durch revolutionäre oder reaktionäre Bewegung fürchten, nur da wird unter ihrem Schutze Handel, Landbau und Gewerbe fröhlich gedeihen und Wohlstand und Wohlbefinden sich überall verbreiten. Von diesem Geiste beseelt, haben es die Unterzeichneten, von verschiedenen Seiten aufgefordert, unternommen, ein täglich erscheinendes Blatt für Politik, Handel und Gewerbe unter dem Titel „Norddeutsche Zeitung" herauszugeben, und hoffen auf zahlreiche gleichgesinnte Mitarbeiter.

„Die ächte Freiheit auf dem Boden der Sittlichkeit und des Gesetzes und in lebendiger, selbstständiger Gliederung" das sei das Loosungswort dieser Zeitung, das sei das Feldgeschrei aller derer, die sich um sie schaaren wollen, und mit ihr den Kampf für Freiheit und Recht, den Kampf wider Revolution und Reaktion bestehen wollen.

Die Norddeutsche Zeitung wird in ihre Spalten aufnehmen: 1. leitende Artikel über Politik und sociales Leben, über Handel, Landbau und Gewerbe, über großartige literarische Erscheinungen der Zeit; 2. politische und Handelsnachrichten, welche die Herausgeber durch außerordentlich getroffene Einrichtungen den Lesern in den östlichen Provinzen gleichzeitig mit den Berliner Zeitungen zu liefern im Stande sind; 3. Anzeigen und Inserate aller Art. Auch wird die Zeitung kürzeren humoristischen Aufsätzen und Gedichten, sofern sie mit den Zeitereignissen in Zusammenhang stehen, gerne ihre Spalten öffnen. Dagegen wird die Norddeutsche Zeitung allen persönlichen Verdächtigungen und Schmähungen ihre Spalten versagen.

Für den Handel wird die Norddeutsche Zeitung im Gegensatze gegen das süddeutsche Schutzzollsystem das Prinzip des **Freihandels** verfechten, jedoch nicht in dem Sinne derjenigen Freihandelsmänner, welche die Freiheit mit dem Mangel aller organischen Gliederung verwechseln, sondern in der Weise, daß auch auf diesem Gebiete, wie überall, jede besondere Organisation, die ihre Berechtigung hat, geschützt und gesichert werde. Dieselben Grundsätze wird sie auch für die Gewerbe vertreten und den Interessen des Ackerbaues und der ländlichen Industrie besondere Aufmerksamkeit widmen.

Die Zeitung wird aus zwei Blättern bestehen, einem politischen und einem Blatte für Handel und Gewerbe; beide werden täglich, zusammen in der Stärke von etwa zwei Bogen erscheinen. Das vierteljährliche Abonnement für beide Blätter zusammen beträgt hiesigen Orts 2 Rthlr. Das politische Blatt wird auch besonders ausgegeben werden zu dem vierteljährlichen Preise von 1 Rthlr. Der Postdebitspreis wird so billig als möglich gestellt werden. Alle Postämter nehmen Bestellungen darauf an.

H. & R. Grassmann.

Fig. 25. Advertisement for the "Norddeutsche Zeitung"

15 February 1850, he gave up on journalism altogether, leaving his brother in charge. Robert Graßmann carried on as the paper's editor-in-chief and publisher until 1854.[219]

This is how Hermann Graßmann told the story in 1853: "Shortly after the political situation had stabilized and my political vision had not fulfilled itself, I began to feel frustrated with politics. Step by step, I took off the political gown, in which I had never felt comfortable. Gradually I began to spend more and more time on mathematics again, until I literally severed my ties to politics. I left the newspaper and everything related to it to my brother and once more dedicated my free time, of which my professional obligations left me very little, to mathematics."[220]

Hermann Graßmann would never make any public statement on politics again.

1.7 Renewed struggles for recognition as a mathematician

In mid-1849, Hermann Graßmann's interest in political action and the newspaper was fading. But there must have been more to this change of heart than political disillusionment and his wish to continue his research in mathematics: In April 1849, Hermann Graßmann married Marie Therese Knappe. He was almost 15 years older than his bride, who was the daughter of Pomeranian landowners. They had become engaged in August 1848.

Hermann Graßmann and his wife Therese, *née* Knappe, had eleven children:

1. Emma Dorothea Johanna (1850 – 1923);
2. Karl Justus (1851 – 1909);
3. Max Siegfried (1852 – 1917);
4. Robert Helmuth (1854 – 1856);
5. Agnes Klara (1855 – 1925);
6. Hermann Ernst (1857 – 1922);
7. Luise (1858 – 1859);
8. Ludolf Edmund (1861 – ?);
9. Karl Richard (1864 – 1938);
10. Klara Marie Therese (1866 – 1881);
11. Konrad Günther (1867 – 1877).[221]

Justus, his oldest son, studied mathematics and natural sciences in Göttingen, Leipzig, Königsberg and Berlin. He finished his dissertation in Berlin in 1875 and became a teacher a year later. In 1901 he became the headmaster of the "Friedrich-Wilhelm-Realgymnasium" in Stettin.

Hermann studied mathematics and natural sciences in Leipzig and Halle. He graduated in 1880. He finished his habilitation in 1899 and became a professor at the University of Gießen in 1904. He dedicated his life to continuing and propagating his father's *Extension Theory*.

Ludolf studied medicine in Berlin. He finished his dissertation in 1885 and became a military doctor. He later returned to Stettin and practiced as an ear, nose and throat specialist.

Finally, **Richard** studied mechanical engineering in Berlin and received a position with the government in 1895. After leaving this official position, he supervised the construction of power plants as an employee of the electric power division of the Berlin-based AEG company. He was responsible for "electric power plants in Baku on the Caspian Sea, Barcelona, Genoa, Buenos Aires and Santiago de Chile..."[222] In 1902 he became a professor at the Polytechnic University of Karlsruhe.

Hermann Graßmann was forty years old. His marriage was a very happy one. He and his wife had eleven children, seven of which were still alive in 1911. Many of his children successfully pursued a career in science. Hermann Graßmann, who had many brothers and sisters, was a family man and his constantly growing family made him very glad. His children loved and respected their father. Thanks to his loving and very supportive wife, Graßmann managed to combine family life with professional obligations and his scientific work.

Friedrich Engel wrote about Graßmann's wife: "His wife made it possible for him to dedicate time, energy and passion to scientific work, in addition to his family and professional obligations. She sincerely loved him, and her willpower and straightforwardness contrasted nicely with his relaxed and forgiving character. She bore the worries of everyday life on her own, and her intelligence, her cheerfulness and practical vein made for a warm and harmonious household. She knew how to keep the children away from the sacred grounds of his study room. But when, for example, it was time for an afternoon cup of coffee, she also knew that it was time for the children to see their father, and he enjoyed their presence ..."[223]

Thanks to his wife, Graßmann's home life permitted him to unfold his enormous creative powers. Therese Graßmann helped her husband to accept the persistent lack of interest he confronted in the scientific community of mathematicians. Therese Graßmann should not be forgotten.

We will now return to Graßmann's research activities. In 1849, after he had turned his back on politics, Graßmann began to prepare the second part of *Extension Theory* for publication. Together with his brother, he had already begun this project in 1847.

Relying on these first drafts, Graßmann assembled the first chapters. But his work was interrupted when he asked O. Wiegand, the publisher of the *Extension Theory* of 1844, whether he was interested in the second part. O. Wiegand's answer was so unsatisfactory that Graßmann decided to stop the project. In May 1853, he wrote to Möbius: "Also, given the fact that the first part of the book received so little attention, it seemed quite useless to me to keep working on its completion, at least in the original form: I now believed that it was better to revise the entire work at a later point in time. I would have to find the simplest form for its principles, which, as I hoped, would also please the mathematicians."[224]

Möbius strongly encouraged Graßmann and urged him to revise *Extension Theory* as quickly as possible.[225] But years would pass before the project was completed in 1861.

In the meantime, Graßmann published articles in which he applied his extension theory to the theory of algebraic curves and surfaces, as he had already done before the revolution of 1848. In July 1850, Graßmann published one article (H. Graßmann 1851c), two in July 1851 (H. Graßmann 1851a, 1851b), one in December 1851 (H. Graßmann 1852) and no fewer than five articles in July 1852 (1855a – 1855e). Graßmann thus hoped to make his findings known to a wider public. But apart from two publications in 1854 and 1855 (H. Graßmann 1855f, 1855g), in which he defended the originality of his work against

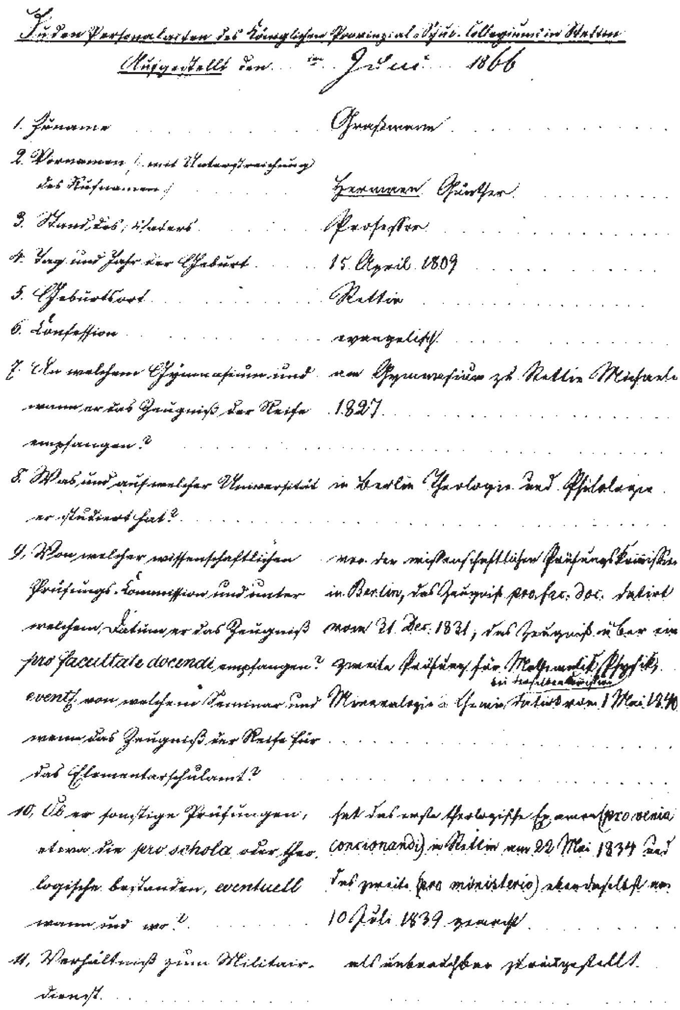


Fig. 26. First page of a registration form filled out by Graßmann for the Stettin "Provinzial-Schul-Collegium", 1 June 1866

Cauchy and Saint-Venant and responded to scientific objections expressed by Bellavitis[226], Graßmann published no further journal articles on mathematics until November 1872.

As we will see, this 17-year pause in Graßmann's publications occurred because, after 1853, he dedicated his energies to the revised *Extension Theory* (A2) and his *Arithmetic* (LA). But after these two books had been published in 1860 and 1861[227] with no success to speak of, Graßmann was disillusioned, gave up on mathematics and focused completely on philology.

Graßmann's renewed interest in scientific research in 1849 was by no means confined to mathematics. Graßmann returned to the Stettin Physics Society, of which he had been a member since 1837.[228] He had not given any lectures since February 1848, but he began lecturing twice a year in 1850. He chose up-to-date topics such as the theory of electricity, optics, acoustics, and selected aspects of chemistry.

Lectures given by Graßmann at the Physics Society were, for example:

- in 1850, a lecture on diamagnetism (which Faraday had discovered in 1845),
- in 1851, an explanation of the just invented Foucault pendulum, which was supposed to furnish proof of the earth's rotation by creating a seemingly rotating pattern,
- in 1862, a lecture on spectrum theory and on Bunsen's spectroscope (Bunsen and Kirchhof 1859),
- in 1863, an explanation of Knoblauch's experiments concerning the diathermancy of rock salt and the wave properties of heat rays (Knoblauch 1848ff.).[229]

After 1849 Graßmann's work for the Physics Society not only gave him important inspiration for his own work but also allowed him to present his own discoveries and inventions in physics to a larger audience.

In 1852 Graßmann lectured on his most recent results in the theory of color shading. A year later, he published his findings in the journal "Poggendorfs Annalen" (H. Graßmann 1853). Graßmann reached a wider audience, and Helmholtz was very impressed with Graßmann's work.

Graßmann pointed out some errors committed by Helmholtz (1852), which Helmholtz then corrected.[230] Drawing conclusions from these errors, Graßmann made Newton's theory of colors more precise (Newton had elaborated his theory in *Opticks: or, a treatise on the reflexions, refractions, inflexions and colours of light*, London 1704). This "emendation" of Newton's theory was quite far-reaching. Graßmann replaced Newton's concept of a discrete spectrum with privileged colors with the concept of a continuous spectrum with no privileged colors, without saying a single word about this fundamental change in Newton's theory. Just by following requirements of his own mathematical theory, he created a new interpretation of the physics of colors.[231]

The treatise *On the Theory of Color-Mixing* ("Zur Theorie der Farbenmischung", H. Graßmann 1853) begins with the following words:
"Mr. Helmholtz [has] published a number of observations, some of which are new and very noteworthy, which have led him to conclude that the essential assumptions of Newton's theory of color-mixing are wrong. According to Mr. Helmholtz, there are only two

prismatic colors, namely yellow and indigo, the mixture of which gives white. But we should point out that Newton's theory of color-mixing is correct up to a certain point: the theorem that every color possesses a spectral complementary which, when the two are mixed, gives white, arises from undeniable mathematical truths. This is one of the most secure theorems of physics. I will show that Helmholtz's *positive* observations, instead of contradicting this theory, can serve to confirm parts of it and complement it."[232]

This is how he formulated his laws of color mixing:

1. "...every color sensation [consists of] ... three variables which can be defined mathematically...: *hue*, *brightness*, and *saturation*."[233]
2. "*...when two lights are mixed and one of the two changes continuously ... then the way we perceive the mixture will also change continuously.*"[234]
3. "*...two colors with constant hue, constant brightness and constant saturation* [will also give] *a constant mixture of colors ..., regardless of which homogeneous colors it is made up*."[235]
4. "*...the total brightness of the mixture* [equals] *the sum ... of the brightness of the colors which have been mixed*."[236]

Graßmann's achievement was to make the Newtonian empirical laws of color theory[237] "... more precise by using the Graßmannian laws as simple theorems, thereby creating a basis for other scientists to build on"[238].

Graßmann had found basic theorems, and he used them to reach his original goal of deriving a rule for color mixing. He showed that colors could be added, just like vectors, and that this form of representation was identical to the Newtonian construction of the center of gravity. Ideas from his *Extension Theory* served as a heuristic for uncovering the laws of nature. He sent a letter to Möbius in which he explained: "Recently I have found an interesting application for the barycentric calculus in the field of optics ... More and more relations are beginning to appear between geometrical analysis and the laws of nature. Then again, this was to be expected, *assuming this analysis was adequate to nature*. [emphasis mine – H.-J. P.]"[239]

Graßmann's laws played an important role in the work of Helmholtz (1867) and Maxwell (1859). Helmholtz used these laws as a basis for constructing his theory of color mixing. Maxwell used Graßmann's law of color mixing to determine the exact position of the spectral colors on the color chart.[240]

In the following years, Graßmann's laws went through many revisions. It turned out that they only applied to a limited range of color brightness. In recent scientific literature, Graßmann's laws have been subsumed under a "basic law of color metrics", according to which "[t]he retinal cone cells, adapted to bright light, linearly and continuously discern incoming light rays in three independent and spectrally distinct functional ef-

Fig. 27. Hermann von Helmholtz (1821 – 1894)

fects. The individual effects are combined in an additive and linear way to form a unitary, inseparable total, which is called *color valency*."[241]

Graßmann's work on the theory of color mixing has had a strong influence on the development of color metrics. The importance of his scientific achievement is underlined by the fact that, 60 years after the publication of Graßmann's treatise, the important German physicist and chemist W. Ostwald made grave mistakes in his color theory, which he considered the most important achievement of his career. These mistakes concerned exactly those problems which Graßmann had clarified.[242] Today, Graßmann's color laws still carry his name.

In November 1854, Graßmann presented his theory on the physics of vowels. In September of that same year, he had published some elements of his theory in an overview of the curriculum of the Stettin "Gymnasium" (H. Graßmann 1854). Needless to say, specialists did not take notice of this first publication and in 1877 – the year of his death – Graßmann republished his theory for a wider readership. Once again, Graßmann and Helmholtz had some points in common.

Graßmann's work on the theory of vowels is important because it is considered an important step towards Helmoltz' theory of vowel resonance (1859ff).[243] Like R. Willis (1832) and Ch. Wheatstone (1837), Graßmann (1854) assumed that vowels were characterized by the fact that certain overtones accompany the fundamental tone especially strongly due to resonance in the mouth cavity.

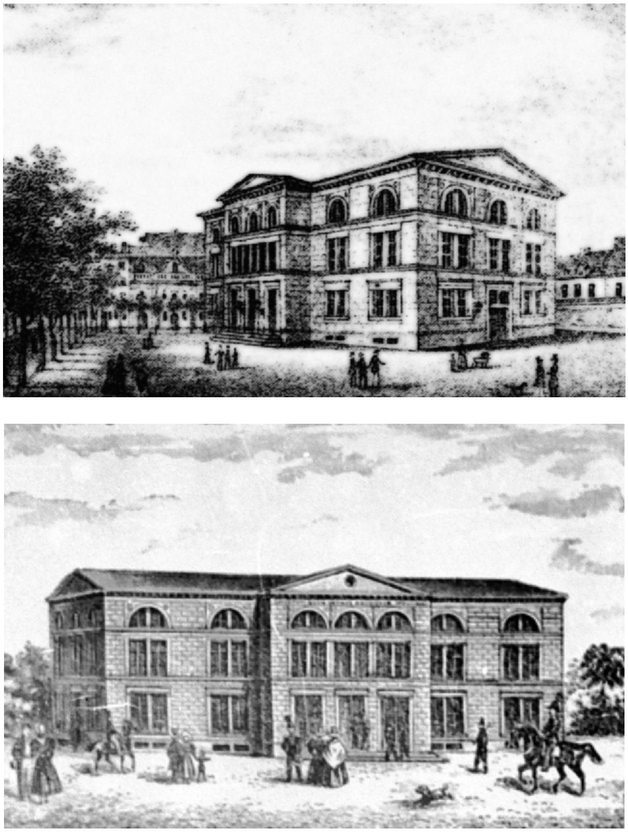


Fig. 28./29. The Stettin "Gymnasium" in Graßmann's day

But Graßmann's practical execution of his approach was flawed. Graßmann attempted to conceptualize the theory of vowels in analogy to the theory of color mixing by constructing centers of gravity in a "vowel plane". "If one represents", Graßmann explained in 1877, "U, I, A, or any other group of three vowels in which one of the three is not situated between the other two [that is, which cannot be sung in a continuous sound along with the other two – H.-J. P.], by three points in a plane, then every other vowel can be represented by one specific point in this plane."[244]

Graßmann committed this error in his theory because, at the time, he could only rely on his exceptionally well-trained hearing, but lacked mechanical resonators and other instruments. Only in the 1920s were the resonance theory of vowels and the analysis of the corresponding oscillation in the vowel sound generally recognized in science.[245]

Despite his misjudgments, this theory made Graßmann one of the founders of the modern theory of vowels.

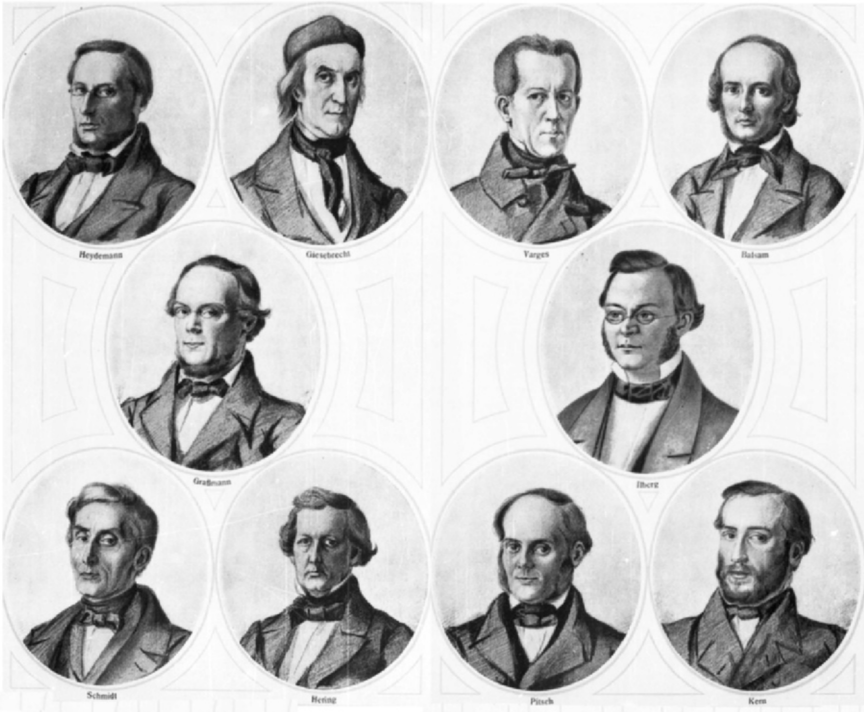


Fig. 30. The teachers of the Stettin "Gymnasium" in the mid-1850s. Pencil drawing by a pupil.

Graßmann also improved the design of the heliostat, which he developed for the Berlin Physics Society. In the 1870s, he had it built by the important Berlin engineer Fuess.

The Stettin Physics Society fell apart after Graßmann's death in 1877. This shows how important Graßmann was for local scientific life.

Apart from his research in mathematics and physics, which was back on Hermann Graßmann's agenda after 1849, philology also became an important aspect of his work. In the years after 1861, disappointed by mathematics, he turned to philology. Not only had he been interested in philology during his student days, he had also begun to lay the foundations for philological research in 1849. These preparations enabled him to move into this new field of research.

General interest and the need to take time off from his "mind-numbing" work in mathematics[246] were also reasons to get involved in the new and still developing science of comparative philology. Inspired by the works of F. Bopp, the founder of historical and comparative philology in Germany, Graßmann began to study Gothic, Lithuanian, Old Prussian, Old Persian, Russian and Church Slavic. Also, he enthusiastically studied Bopp's comparative grammar (Bopp 1833 – 1852). By the late 1850s, Graßmann was obtaining the first results from his work on this important group of Indo-European languages. Philologists immediately took note of the articles which Graßmann published between 1860 and 1863.[247] Apart from his later work on the *Rig-Veda*, Graßmann was noticed for his law of aspiration, which he published in a journal in 1863 (H. Graßmann 1863). The treatise on aspirates "made Graßmann's name an indelible part of the history of philology."[248] He had made an important contribution to understanding the Germanic sound shift.[249] Graßmann was over forty years old when he became interested in philology and over fifty years old before he obtained results in a field that had nothing to do with mathematics. His achievements were so noteworthy that he received an honorary doctoral degree from the University of Tübingen in 1876.[250] These facts remind us of Graßmann's creativity and energy as a scientist, which, with his total commitment to his projects, led to these great achievements.

One can hardly imagine how Graßmann managed to work on such a wide variety of scientific topics: in 1849/50, in school, he even had to stand in for a colleague who had fallen ill for an extended period of time.[251]

However, his new start in science was clouded by other problems: On 9 March 1852, his father died. This meant that the Stettin "Gymnasium" had lost its professor of mathematics, and the Stettin Physics Society its founder and president.

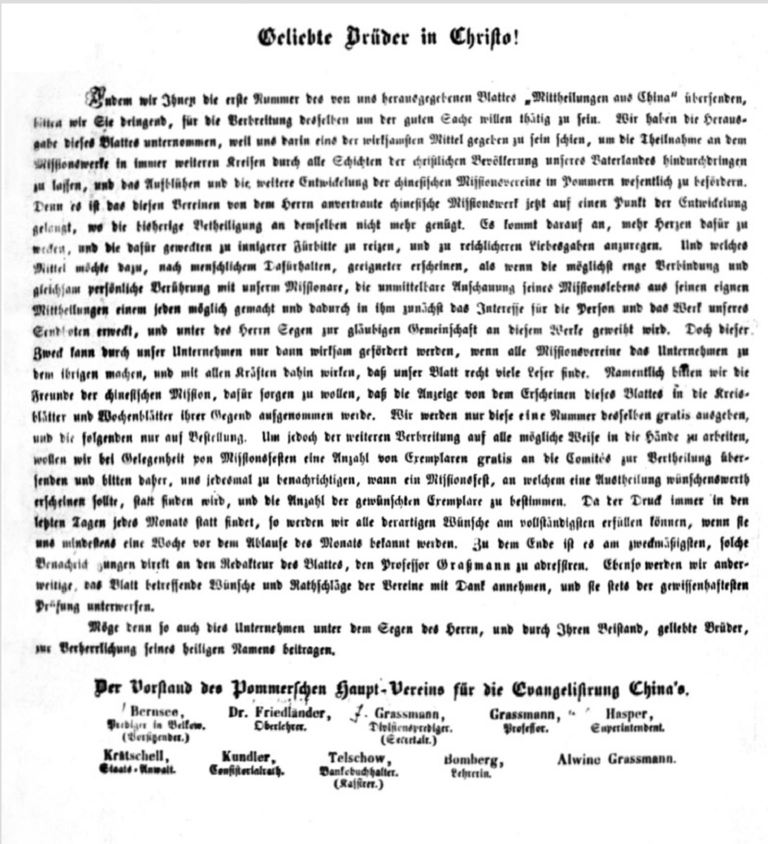
That same year, Hermann Graßmann, whose way of thinking and scientific interests were very similar to his father's, became his successor at the Stettin "Gymnasium" and in the Physics Society.

Graßmann's financial situation had improved: While he had only earned a yearly income of 800 taler at the "Friedrich-Wilhelmsschule", it had now risen to 1250 taler. But this also meant that a position at a German university was hardly an option anymore. Most university professors earned much less.

Graßmann's new position affected his scientific studies adversely. While mathematics and physics had not been at the center of Graßmann's work at the "Friedrich-Wilhelmsschule", he now had to teach mathematics and physics almost full-time. Graßmann had to spend more time preparing his classes. But he had moved up in the hierarchy of teachers and received the title of Professor, which so far he had been denied.

Graßmann spent the rest of his life teaching 18 to 20 hours a week. In 1853, he became the conductor of the student choir, which gained in popularity and was one of Graßmann's favorite projects. The pupils called Graßmann "the pastor" because of his humble and religious temperament. The school choir became known in Stettin as "the pastor's club".[252]

Graßmann became a member of the **Pomeranian Society for the Evangelization of China** in 1857. This organization, which had been founded in 1850, aimed to train Protestant missionaries in order to convert the Chinese. Hermann Graßmann made first financial contributions to the organization in 1853. His younger brother Justus Gotthold Graßmann, a military clergyman in Stettin, had been the club's secretary since 1852. When, in 1858, Justus Gotthold Graßmann was transferred to a different location, Hermann Graßmann became a member of the club's organizing committee and replaced his brother. In the 1870s, he became the director of the Pomeranian Society. Apart from his many other obligations, around Easter 1858, Graßmann also began to help with the publication of the club's journal "Reports from China" (1858 – 61). Since the "evangelization of China" made little progress, publication of the journal ended in 1861.[253]



Geliebte Brüder in Christo!

Indem wir Ihnen die erste Nummer des von uns herausgegebenen Blattes „Mittheilungen aus China" übersenden, bitten wir Sie dringend, für die Verbreitung desselben um der guten Sache willen thätig zu sein. Wir haben die Herausgabe dieses Blattes unternommen, weil uns darin eins der wirksamsten Mittel gegeben zu sein schien, um die Theilnahme an dem Missionswerke in immer weiteren Kreisen durch alle Schichten der christlichen Bevölkerung unseres Vaterlandes hindurchdringen zu lassen, und das Aufblühen und die weitere Entwickelung der chinesischen Missionsvereine in Pommern wesentlich zu befördern. Denn es ist das diesen Vereinen von dem Herrn anvertraute chinesische Missionswerk jetzt auf einen Punkt der Entwickelung gelangt, wo die bisherige Betheiligung an demselben nicht mehr genügt. Es kommt darauf an, mehr Herzen dafür zu wecken, und die dafür geweckten zu innigerer Fürbitte zu reizen, und zu reichlicheren Liebesgaben anzuregen. Und welches Mittel möchte dazu, nach menschlichem Dafürhalten, geeigneter erscheinen, als wenn die möglichst enge Verbindung und gleichsam persönliche Berührung mit unserm Missionare, die unmittelbare Anschauung seines Missionslebens aus seinen eignen Mittheilungen einem jeden möglich gemacht und dadurch in ihm zunächst das Interesse für die Person und das Werk unseres Sendboten erweckt, und unter des Herrn Segen zur gläubigen Gemeinschaft an diesem Werke geweiht wird. Doch dieser Zweck kann durch unser Unternehmen nur dann wirksam gefördert werden, wenn alle Missionsvereine das Unternehmen zu dem ihrigen machen, und mit allen Kräften dahin wirken, daß unser Blatt recht viele Leser finde. Namentlich bitten wir die Freunde der chinesischen Mission, dafür sorgen zu wollen, daß die Anzeige von dem Erscheinen dieses Blattes in die Kreisblätter und Wochenblätter ihrer Gegend aufgenommen werde. Wir werden nur diese eine Nummer desselben gratis ausgeben, und die folgenden nur auf Bestellung. Um jedoch der weiteren Verbreitung auf alle mögliche Weise in die Hände zu arbeiten, wollen wir bei Gelegenheit von Missionsfesten eine Anzahl von Exemplaren gratis an die Comités zur Vertheilung übersenden und bitten daher, uns jedesmal zu benachrichtigen, wann ein Missionsfest, an welchem eine Austheilung wünschenswerth erscheinen sollte, statt finden wird, und die Anzahl der gewünschten Exemplare zu bestimmen. Da der Druck immer in den letzten Tagen jedes Monats statt findet, so werden wir alle derartigen Wünsche am vollständigsten erfüllen können, wenn sie uns mindestens eine Woche vor dem Ablaufe des Monats bekannt werden. Zu dem Ende ist es am zweckmäßigsten, solche Benachrichtigungen direkt an den Redakteur des Blattes, den Professor Graßmann zu adressiren. Ebenso werden wir anderweitige, das Blatt betreffende Wünsche und Rathschläge der Vereine mit Dank annehmen, und sie stets der gewissenhaftesten Prüfung unterwerfen.

Möge denn so auch dies Unternehmen unter dem Segen des Herrn, und durch Ihren Beistand, geliebte Brüder, zur Verherrlichung seines heiligen Namens beitragen.

Der Vorstand des Pommerschen Haupt-Vereins für die Evangelisirung China's.

Bernsee, Prediger in Belkow. (Vorsitzender.) — Dr. Friedländer, Oberlehrer. — J. Grassmann, Divisionsprediger. (Secretair.) — Grassmann, Professor. — Hasper, Superintendent.

Krätschell, Staats-Anwalt. — Kundler, Consistorialrath. — Telschow, Bankobuchhalter. (Kassirer.) — Bomberg, Lehrerin. — Alwine Grassmann.

Fig. 31. Leaflet accompanying the first issue of the missionary journal "Reports from China" ("Mittheilungen aus China"). Professor Graßmann was among the organizers of the project.

This is how Graßmann's former pupils remembered their teacher

A. Müller:
"Graßmann was an infinitely benevolent person and extremely patient with others. He was very good at dealing with the somewhat mischievous youngsters."[254]

G. Wandel:
"He was an optimist, always on time, always friendly, very forgiving and patient, hardly ever angry when pupils wore down his patience, nothing vindictive in his nature..."[255]

A. Jonas:
"It happened frequently that ... in Graßmann's classes pupils did the homework for other subjects. He could not bring himself to intervene and reprimand them sharply. ...I remember the following incident very well. Some up-to-no-good lads were violently disrupting Graßmann's class, and they paid no attention to his reprimands. Instead of making use of what was *ultima ratio magistrorum* even in higher grades, Graßmann suddenly jumped on his podium and said a long and loud prayer. He pleaded to God not to count their rowdy behavior as a sin and to enlighten them with His mercy."[256]

P. Rusch
"I remember that, in ninth grade, he was our teacher of religion for a semester. He would pray at the beginning and at the end of each class. One day a pupil had annoyed him very much and, when class was almost over, he said that he could not pray because he was too angry. This made a great impression on us, and when the offender approached Graßmann after class to apologize, Graßmann forgave him and kissed him."[257]

M. Wehrmann:
"His classes in mathematics were quite unusual. Graßmann mostly paid attention to the pupils who were especially interested in mathematics and had a gift for it. The others misbehaved most of the time."[258]

Finally, in the mid-1850s, Graßmann also became a leading member of the Pomeranian Society for the Evangelization of China and helped publish its little leaflet in 1858.

His many new obligations forced Hermann Graßmann to postpone his plans for revising *Extension Theory*. He did not even get around to renewing the contact to Möbius, which had been interrupted in the mid-1840s and which Graßmann valued very highly.

Graßmann was far less successful as a teacher than he was as a scientist.[259] He lacked the necessary authority. His profoundly religious attitude, his straightforward behavior and the calmness of a man of science only gave him a certain degree of authority, which few pupils felt. All the others misbehaved during his classes, which were not terribly successful from an educational point of view. Already during Graßmann's education at the seminar for teachers, the Provincial School Authority of Brandenburg had pointed to the "peculiarly timid personality of Mr. Graßmann"[260], which had given rise to doubts concerning his aptness for the job. In his first position as a teacher of mathematics at the Berlin School of Commerce

(1834/35), there had been considerable problems with discipline in Graßmann's classes. In Graßmann's file, the school's headmaster remarked: "Discipline has always been one of the strong points of this school. But in the last six months, discipline in mathematics classes has waned, despite my careful and energetic support. Pupils accustomed themselves to this lack of discipline, and it even began to spread to other classes ..."[261]

In later years, many of Graßmann's pupils remembered him as a man who often left the impression that his mind was on something else.[262] As a teacher, he represented a quite unpopular subject. As a person, he was idealistic and benevolent.

Only on 22 May 1853 did Graßmann manage to resume his correspondence with Möbius, who by then was 63 years old: "As soon as I had returned to my studies in mathematics, I also felt that I wanted to renew our intellectual contact. In literally all of my mathematical work and ideas, almost miraculously, I have always encountered you, one who understands my approach and for whom I have great esteem", Graßmann wrote in his first letter to Möbius. But Graßmann's project had been interrupted by the many happy and tragic events mentioned above. "My old father's illness and, finally, his death on 9 March interrupted my work and projects and obliged me to think of other things. When I acceded to my father's position at the 'Gymnasium', I was very busy preparing classes in mathematics and physics, which so far I had only taught sporadically. This explains why it has taken me until now, our Pentecost vacations, to carry out my plan."[263]

This would prove to be a highly valuable and inspirational exchange of ideas for Graßmann.

Graßmann informed Möbius about the details of his 1853 theory of color mixing and about his 1854 research on the theory of vowels. He told Möbius that he was thinking about an apparatus resembling a harmonium which would serve to verify his theory experimentally. He and Möbius exchanged thoughts on musical harmony.[264] Möbius was very interested in discussing all of these questions. He pointed out new ideas, criticized Graßmann and shared results from his own work.

But the two men mostly discussed mathematical problems. They exchanged recent work, and Möbius was very supportive of Graßmann's mathematical projects.

In his first letter to Graßmann after 1848, Möbius immediately pointed out that, in 1853, Cauchy and Saint-Venant[265] had published articles in the "Comptes Rendus" which stated exactly what Graßmann had said in § 45 and § 93 of his *Extension Theory* of 1844. "I certainly hope", Möbius wrote, "that you will assert your priority of authorship against Cauchy and Saint-Venant. And I also hope that you will dedicate your entire energy to completing the *revised* and *AMPLIFIED version* of your Extension Theory."[266]

This question of scientific priority was a decisive impulse for Graßmann: he began to tackle the new version of *Extension Theory*. In February 1854, referring to Cauchy's and Saint-Venant's articles, he wrote Möbius: "This whole matter has prompted me to begin with the new version of my *Extension Theory* ... I am planning to leave out all of its

Fig. 32. August Ferdinand Möbius (1790 – 1868)

complicated applications to mechanics and physics and to keep the entire structure as simple as possible. I will also skip all difficult and abstract sections as well as those which I have not mastered completely. The new book should not run the risk of being ignored again, or of only being studied thoroughly by a single reader."[267]

In January 1855 he informed Möbius that he was planning to have his revised *Extension Theory* published that summer.[268] In May he told Möbius that he was spending every free minute completing the new book. Nevertheless, he would only complete his second great mathematical masterpiece in October 1861.

In the meantime, the debate with Cauchy and Saint-Venant was underway.[269] Graßmann had already encountered the work of these two mathematicians before 1853. In 1847, he had found an article by Saint-Venant in the "Comptes Rendus" of September 1845. In this tract, Saint-Venant had developed the geometrical addition and multiplication of displacements. Since Graßmann did not have Saint-Venant's address, he sent a letter to Cauchy, along with two copies of his *Extension Theory* and a letter to Saint-Venant pointing out that he had already discovered these structures some time ago. He asked Cauchy to pass the letter and a copy of *Extension Theory* on to Saint-Venant.

Saint-Venant responded that *Extension Theory* had not arrived but that he was very interested in reading it. In January 1848, Graßmann wrote a long answer to Saint-Ve-

Fig. 33. Augustin Louis Cauchy (1789 – 1857)

nant and also sent him his prize-winning treatise (PREIS) and his *On a Purely Geometrical Theory of Curves* ("Grundzüge einer rein geometrischen Theorie der Kurven", H. Graßmann 1846).

After this, Saint-Venant stopped communicating with Graßmann. A translation of Graßmann's *Geometrical Analysis* was found in Saint-Venant's library after his death, which proves his interest in the work of Graßmann.

In 1853 Graßmann was confronted with two treatises which coincided almost completely with his findings in *Extension Theory*. Since these treatises had been written by the only two French mathematicians to whom he had sent his work, he was quite annoyed. In early February 1854 he wrote an article in French for "Crelles Journal" which appeared in 1855, *Sur les différents genres de multiplication* (H. Graßmann 1855f). In the article, Graßmann pointed to the fact that Cauchy's theory did not differ from what he had already published in 1844. Graßmann also elaborated on some questions of algebra. In April 1854, Graßmann also sent an official letter to the Academy in Paris in which he claimed his rights of authorship. The Academy formed a commission to investigate Graßmann's objection. Lamé, Binet and Cauchy himself were on it. The commission never responded to Graßmann's claims.

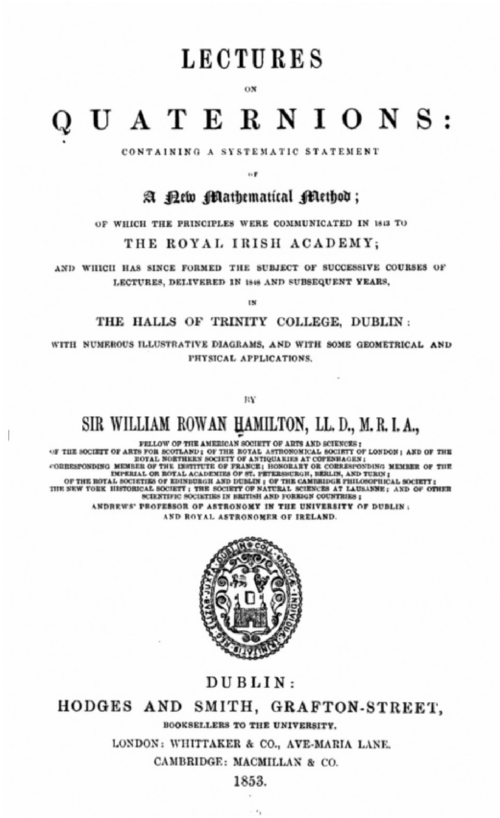
Fig. 34. Sir William Rowan Hamilton (1805 – 1865)

It would be inappropriate to accuse Cauchy of a conscious act of plagiarism. Graßmann also never raised this accusation. Cauchy was an extraordinarily productive and original mathematician, and he might have simultaneously developed the same ideas as Graßmann. But Engel was right when he remarked: "It is inexplicable and unacceptable that Cauchy never reacted to Graßmann's concerns."[270]

In 1853 Graßmann had once again become painfully aware of the fact that his mathematical works had remained unknown in Germany and France, but an Englishman had discovered his writings and could hardly contain his enthusiasm.

The brilliant Irish mathematician and physicist W. R. Hamilton had stumbled upon Graßmann's *Extension Theory* while doing historical research for the introduction to his *Lectures on Quaternions* (Hamilton 1853). He immediately recognized how closely related his work on quaternions was to Graßmann's book. A letter from Hamilton to his friend De Morgan, dated 31 January 1853, shows how impressed he was by his first encounter with Graßmann's work: "I have recently been *reading* … more than a hundred pages of *Graßmann's Ausdehnungslehre*, with great admiration and interest. Previously I had only the most slight and general knowledge of the book, and thought that it would require me to learn to *smoke* in order to read it. If I could hope to be put in rivalship with Des Cartes on the one hand, and with Graßmann on the other, my scientific ambition would be fulfilled!"[271]

Despite his initial enthusiasm, Hamilton's admiration for Graßmann lost some of its force after he had studied the text more closely. Hamilton did not hide his satis-


LECTURES

ON

QUATERNIONS:

CONTAINING A SYSTEMATIC STATEMENT

OF

A New Mathematical Method;

OF WHICH THE PRINCIPLES WERE COMMUNICATED IN 1843 TO

THE ROYAL IRISH ACADEMY;

AND WHICH HAS SINCE FORMED THE SUBJECT OF SUCCESSIVE COURSES OF LECTURES, DELIVERED IN 1848 AND SUBSEQUENT YEARS,

IN

THE HALLS OF TRINITY COLLEGE, DUBLIN:

WITH NUMEROUS ILLUSTRATIVE DIAGRAMS, AND WITH SOME GEOMETRICAL AND PHYSICAL APPLICATIONS.

BY

SIR WILLIAM ROWAN HAMILTON, LL. D., M. R. I. A.,

FELLOW OF THE AMERICAN SOCIETY OF ARTS AND SCIENCES; OF THE SOCIETY OF ARTS FOR SCOTLAND; OF THE ROYAL ASTRONOMICAL SOCIETY OF LONDON; AND OF THE ROYAL NORTHERN SOCIETY OF ANTIQUARIES AT COPENHAGEN; CORRESPONDING MEMBER OF THE INSTITUTE OF FRANCE; HONORARY OR CORRESPONDING MEMBER OF THE IMPERIAL OR ROYAL ACADEMIES OF ST. PETERSBURGH, BERLIN, AND TURIN; OF THE ROYAL SOCIETIES OF EDINBURGH AND DUBLIN; OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY; THE NEW YORK HISTORICAL SOCIETY; THE SOCIETY OF NATURAL SCIENCES AT LAUSANNE; AND OF OTHER SCIENTIFIC SOCIETIES IN BRITISH AND FOREIGN COUNTRIES; ANDREWS' PROFESSOR OF ASTRONOMY IN THE UNIVERSITY OF DUBLIN; AND ROYAL ASTRONOMER OF IRELAND.

DUBLIN:

HODGES AND SMITH, GRAFTON-STREET,

BOOKSELLERS TO THE UNIVERSITY.

LONDON: WHITTAKER & CO., AVE-MARIA LANE.

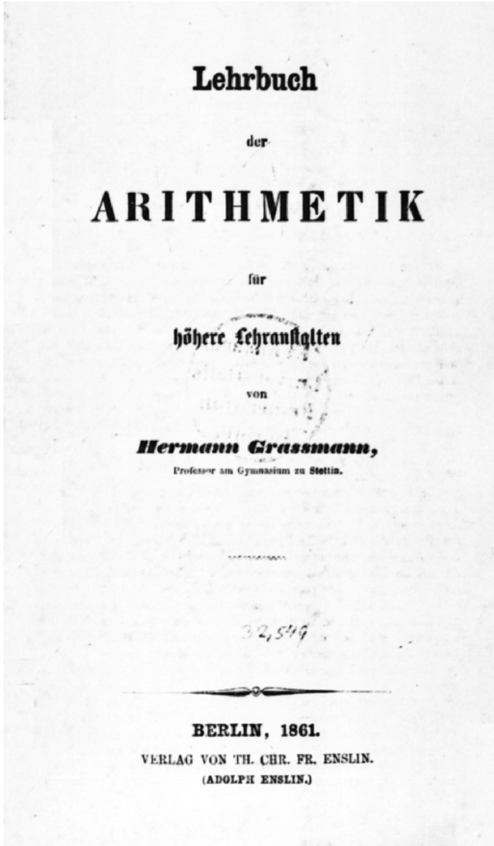
CAMBRIDGE: MACMILLAN & CO.

1853.

Fig. 35. Hamilton's *Lectures on Quaternions*

faction that Graßmann had not discovered quaternions: "Graßmann is a great and most German genius; his view of *space* is as least as new and comprehensive as mine of time; but he has not anticipated, nor attained, the conception of the *quaternions* ..."[272], Hamilton – carefully – praised Graßmann, who in Germany had no readers. It is unfortunate that Graßmann never saw Hamilton's *Lectures on Quaternions* (1853). In the preface, on page 62, Hamilton mentioned his German colleague and expressed his admiration for Graßmann's scientific achievements. It could have soothed Graßmann's disappointment.

It is interesting and quite ironic that Hamilton and Graßmann both overestimated the importance of their mathematical discoveries and developments and, in doing so, increasingly isolated themselves from the mathematics of their time. Hamilton believed that the discovery of quaternions was "... as important for the middle of the nineteenth century as the discovery of fluxions (the calculus) ... for the close of the seventeenth."[273]


Lehrbuch

der

ARITHMETIK

für

höhere Lehranstalten

von

Hermann Grassmann,

Professor am Gymnasium zu Stettin.

32,549

BERLIN, 1861.

VERLAG VON TH. CHR. FR. ENSLIN.

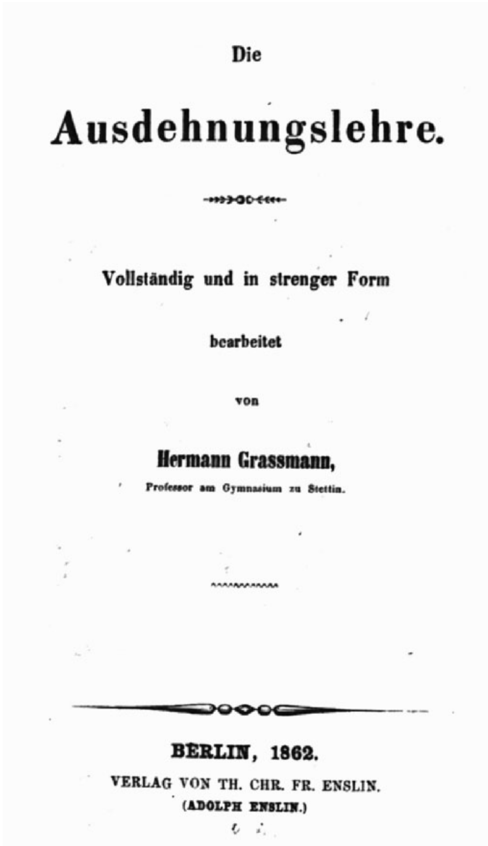
(ADOLPH ENSLIN.)

Fig. 36. Title page of Hermann Graßmann's *Arithmetic*

He dedicated 22 years of his life to research on quaternions. In turn, Graßmann believed that the entire structure of modern geometry, including the theory of invariants, was encompassed by his *Extension Theory*.

Without knowledge of Hamilton's *Lectures*, Graßmann continued his mathematical work. In 1854/55, Hermann and Robert Graßmann resumed their collaboration.[274] Hermann Graßmann summarized the results from this period of time in the *Textbook of Arithmetic* ("Lehrbuch der Arithmetik", LA) of 1860 and the revised version of *Extension Theory* (A2), which he completed in 1861.

In 1855, during a period of intense mathematical research, Graßmann's ambition to become a professor of mathematics awoke once more. At the time, the Berlin Trade Institute (which later became the "Technische Hochschule") was searching for a professor of mathematics. Graßmann immediately asked a brother-in-law who lived in Berlin to


Die

Ausdehnungslehre.

Vollständig und in strenger Form

bearbeitet

von

Hermann Grassmann,

Professor am Gymnasium zu Stettin.

BERLIN, 1862.

VERLAG VON TH. CHR. FR. ENSLIN.

(ADOLPH ENSLIN.)

Fig. 37. The revised version of *Extension Theory*

inquire about the job. But since Graßmann took no official steps to apply and nobody suggested him as a candidate for the job, the dream quickly dissolved.[275]

Graßmann's *Textbook of Arithmetic for Secondary Schools* (LA) is among his most important works. Not only is it remarkable because it appeared before the new version of *Extension Theory*, but also because its form of presentation served as a model for the second *Extension Theory*. The *Textbook of Arithmetik* is also a clear expression of Robert Graßmann's intellectual influence on his brother Hermann. In the introduction, Hermann Graßmann wrote that collaboration with his brother had been an important factor in the genesis of the book. Even the introduction's somewhat arrogant tone points to Robert's influence. One does not find it in Hermann Graßmann's other texts: "The present treatment of arithmetic … brings forward the conviction that it represents the first strictly scientific discussion of this discipline, with the even more far-reaching requirement that its method, however far it may stray from the common paths, should not be seen as one among many, but as the only possible option for a strictly coherent and adequate discussion of this discipline."[276]

Compared to the *Extension Theory* of 1844, this book used a quite different mode of presentation. In earlier works, Graßmann had developed mathematical ideas from concepts. This procedure had made some of his conceptions quite imprecise. Now, in the *Textbook of Arithmetic*, he rigorously applied the Euclidean mode of presentation, with conceptual abstractions – even though the introduction emphasized their importance – disappearing almost completely. The book presents a string of definitions, theorems and proofs. Graßmann contextualized every new conclusion by referring to preceding theorems. The book was not suited for teaching purposes in secondary schools, but it was impressively rigorous and concise.

Influenced by his brother, Hermann Graßmann stopped viewing mathematics as a theory of forms and returned to mathematics as a theory of quantity in the Leibnizian line of thinking.[277] The scientific importance of the *Textbook of Arithmetic* is linked to the fact that the book relies on insights from the *Extension Theory* of 1844, especially its general theory of forms.[278] In the first paragraphs, Graßmann presented a foundation of arithmetic which was highly influential for subsequent research in mathematics.

The "completely revised and rigorously structured" *Extension Theory* (A2) brought this period of mathematical work to a brilliant conclusion.

Hermann Graßmann had 300 copies of the book printed at his own expense in his brother's printing shop and distributed them via the publisher T. C. Enslin. The *Extension Theory* of 1862 was a great step forward compared to its predecessor of 1844. The range of its content was more diverse, mathematical concepts were more precise, and Graßmann had given up on the unrealistic goal of remaining completely independent of analysis. At the same time, the book had a decisive and grave disadvantage: its strictly Euclidean form.

Robert Graßmann had been in favor of this change in the *Extension Theory's* formal structure, but Hermann had also received a letter from the mathematician Grunert in which he had suggested the Euclidean form.[279]

This was a methodological mistake which was prefigured in the *Textbook of Arithmetic*. Graßmann jumped from one extreme to the other. On the one hand, mathematicians could no longer accuse him of being overly philosophical. On the other, Graßmann's very exotic approach now appeared in the most inaccessible mode of presentation there was. Readers could not fathom the potential usefulness of Graßmann's developments. There was absolutely no reaction to the publication of the book. Scholars ignored it, and its impact on the scientific community of mathematics was even smaller than in 1844. But Graßmann placed all his hopes in these two publications. He immediately sent them to the Prussian minister of culture and education von Bethmann-Hollweg, along with the following note:

"Your Excellency,

please allow me to humbly present to you these mathematical works. I also kindly ask you to consider me a candidate should a professorship in mathematics become available at a Prussian university."[280]

Graßmann received the following answer on 11 February 1862: "I will gladly take your wish to become a university professor into account, should the occasion present itself. But I would like to remind you that university professors normally are paid considerably less than what you are presently earning."[281]

This was an elegant way of saying no. For the third time, 53 year-old Graßmann was being denied the possibility to free himself from the heavy workload of secondary-school teaching in order to commit himself completely to his mathematical ideas.

After this setback, firmly convinced of the importance of his ideas for the future development of mathematics, Graßmann completely turned his back on his mathematical research and focused exclusively on philology. Philologists welcomed him with open arms. Graßmann had not found acclaim in mathematics, which was partly his own fault, but his success in philology would make up for past disappointments.

Graßmann withdraws from mathematical research

"And therefore, ever since the publication of my second *Extension Theory*, I have published nothing on mathematics except a textbook on trigonometry (1865). I have kept my silence even when, which has occurred quite frequently, I have read about new insights which are already in my works, completely elaborated and usually in a more complete and extensive manner."[282]

Letter from H. Graßmann to H. Hankel, 8 December 1866

1.8 Farewell to mathematics, success as a philologist and late acclaim in mathematics

After the second version of *Extension Theory* had met with complete silence, Graßmann no longer had the energy to publish further articles and hope for recognition. "In my profession [as a school teacher – H.-J. P.] it is my duty", Graßmann wrote to H. Hankel in late 1866, "to avoid efforts which, if they do not provoke fruitful discussions and do not create the smallest intellectual community, lead to loneliness and provoke a feeling of fatigue, I might even say, frustration. … Since then my independent and serious projects have been in a different field, namely comparative philology …"[283]

Nevertheless, particular circumstances in the early 1870s would lead Graßmann – who by then was over 60 years old – back to mathematics. These circumstances will be discussed below.

But first, Graßmann focused his entire attention on philology, which up to this point he had only used to clear his mind from the strain of mathematical thinking. In the early 1860s, Graßmann's results in historical and comparative philology found great acclaim among philologists. Graßmann felt that he had been successful in getting a new start and he spent many happy hours on his new projects.

We already know that Graßmann was very meticulous in everything he did, and his study projects followed an elaborate plan. He always strove to find the most general structure of a problem and always chose a unitary underlying principle as his point of departure. This also was the case in his philological research, in which Graßmann turned to the oldest sources of the Indo-European languages. Sanskrit was considered one of the most original and authentic languages of the Indo-European family and philologists had begun to concentrate on the oldest documents of Indian literary history, the Vedas. Hermann Graßmann chose the oldest of these texts, the *Rig-Veda*, as his object of study.

When, in the early 1860s, Graßmann began to translate these almost 3000-year-old Indian hymns, he had hardly any scientific literature to rely on. Gradually he received useful publications from other researchers.[284]

Graßmann's will to concentrate on the smallest details and his extraordinary commitment to the project were extraordinary. This is how a former pupil of Graßmann's described this period: "Except for the time he spent with his family, he dedicated every free minute to his work. Assiduously, in his fine handwriting, he wrote word after word, number after number in those enormous collections of texts which gave us his dictionary and translation of the Veda."[285]

Total commitment

"He never went out in the evening to have a glass of beer. When in 1863 the German Association of Natural Scientists met in Stettin and Virchow asked Graßmann to show them a beer tavern, he did not know where to take them."[286]

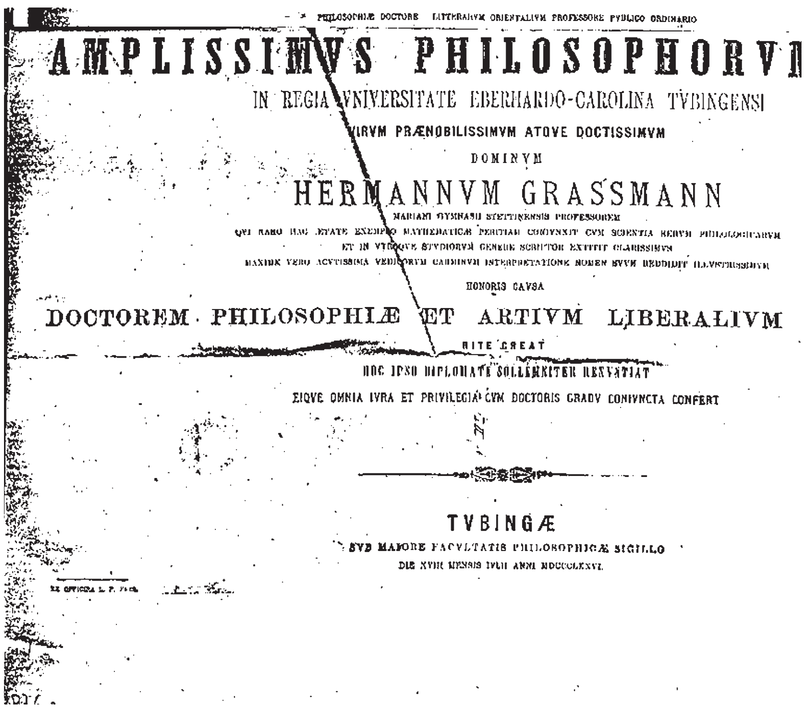
After ten years of hard work, which took up literally all of his free time[287], he finished the manuscript for his *Dictionary of the Rig-Veda* (H. Graßmann 1873 – 75) in 1872. It was published in six parts between 1873 and 1875 by F. A. Brockhaus in Leipzig.

It found immediate acclaim among Sanskrit specialists. Respected Veda researchers such as T. Benefey[288] gave the book excellent reviews. The American Oriental Society made Graßmann a member in 1876, and on 13 July of that same year, the University of Tübingen granted him the title of *Doctor philosophiae et artium liberalium magister honoris causa*, following the suggestion of the important philologist R. Roth.[289] Graßmann was finally receiving the recognition which the scientific community had persistently denied him in mathematics.

Although many passages of Graßmann's work on the *Rig-Veda* quickly became outdated due to exceptional progress in Vedaic research, which soon developed into true "Vedology"[290], Graßmann nevertheless remains an important figure in this field of philology. His achievements, brilliant at the time, guaranteed him a position among the most respected Veda researchers of the late 19th century.

Graßmann's dictionary is still in use today as an important reference. It was reprinted for the sixth time in 1996 (H. Graßmann 1996).

Vedaic research was not Graßmann's only philological project in the 1860s. Together with his brother Robert and his brother-in-law C. Heß, Graßmann worked on a book called *German Plant Names* ("Deutsche Pflanzennamen", H. Graßmann 1870). The process of publishing it extended from 1867 to 1870. This was a work of critical



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ET IN VTROQVE STVDIORVM GENERE SCRIPTOR EXTITIT CLARISSIMVS

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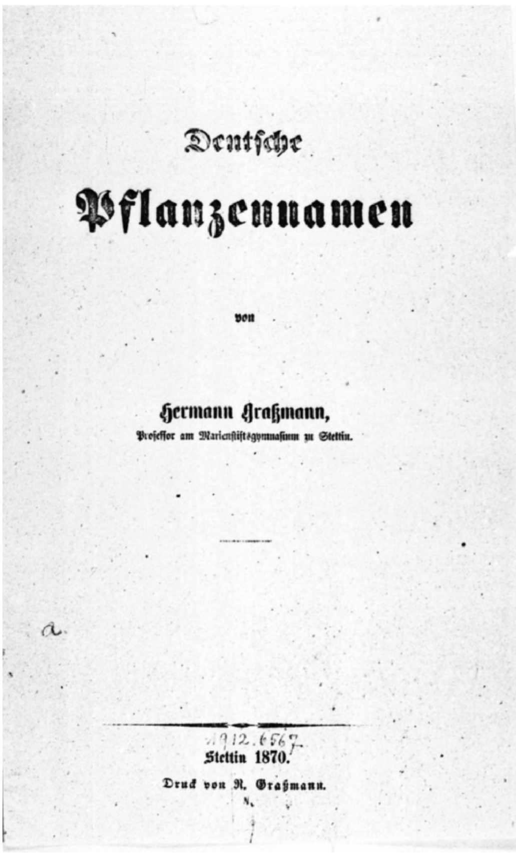
Fig. 38. Graßmann's honorary doctorate from the University of Tübingen, dated 18 July 1876

comparative philology. Graßmann mined 56 works on botany to compile a book of "scientifically established German designations"[291].

"Every science aims to become a popular science. A science can only reach this goal if it also uses a popular language."[292]

"… the introduction of German *designations* is important for science, if it is not to remain confined to small circles of people who have learned these two languages [Greek and Latin – H.-J. P.] and, in this isolation, become an inanimate formalism [emphasis mine – H.-J. P.]."[293]

This is how Graßmann justified the need for his tract, apart from his project of supporting the subject of botany in elementary schools and schools for girls. The book was part of the Graßmann brothers' Germanophile agenda.


Deutsche

Pflanzennamen

von

Hermann Graßmann,

Professor am Marienstiftsgymnasium zu Stettin.

Stettin 1870.

Druck von R. Graßmann.

Fig. 39. Title page of *German Plant Names* by Hermann Graßmann

Even though this throws a shadow on Graßmann's reasons for writing the book, it still does not say anything about the book's philological value. Philologists praised Graßmann's work on the etymology and linguistic structure of German plant names.[294] A number of textbooks for elementary schools relied on Graßmann's insights. But the book did not usher in purely German biological nomenclature for plants.

While Graßmann found satisfaction in his philological studies, he also realized in November 1866 that his mathematical work was finally beginning to have an impact.

Graßmann received a letter from the young mathematician Hermann Hankel, who unfortunately died at age 34. This letter renewed Graßmann's old wish that someone might connect his thinking to recent developments in mathematics "and thereby make a positive contribution to their further development"[295].

In the 1860s, Hankel, an ambitious pupil of Riemann, wanted to publish a systematic textbook on Riemann's theory of the functions of complex magnitudes. While working on the first and regrettably only volume of his *Lectures on Complex Numbers and their Functions* ("Vorlesungen über die komplexen Zahlen und ihre Funktionen"), which bore the title *Theory of Complex Number Systems* ("Theorie der komplexen Zahlensysteme", Hankel 1867), Hankel had turned to Hamilton's *Lectures on Quaternions* to study the characteristics of complex and hypercomplex numbers and had encountered Hamilton's admiring comment on Graßmann's achievements. Hankel decided to study *Extension Theory* and was electrified by Graßmann's ideas.

On 24 November 1866, Hankel wrote Graßmann: "After having read them [Hamilton's *Lectures on Quaternions* – H.-J. P.], my attention was drawn to both of your 'Extension Theories'. To my delight, I realized that these two books treat and use the concept of complex numbers – this is how I designate your extensive magnitudes – so adequately and universally that they helped me a great deal in understanding the matter I was concerned with."[296]

Unlike many German mathematicians before him, Hankel was not dissuaded from studying *Extension Theory* by its mode of presentation and high level of abstractness. This had to do with at least four different reasons:

Firstly, Hamilton's *Lectures on Quaternions* had introduced Hermann Hankel to Graßmann's works. At the time, Hamilton was quite unknown in Germany. In Great Britain, however, he was a much respected mathematician, and he was knighted for his scientific achievements.[297] Hamilton's and Graßmann's theories had many points in common, and therefore Hankel was well-prepared to study *Extension Theory* and quickly became familiar with Graßmann's concepts.

Secondly, since childhood, Hankel had been especially interested the history and philosophy of mathematics. His general mindset was close to Graßmann's, and Hankel was fascinated, not discouraged, by the combination of philosophy and mathematics in the first *Extension Theory*.

Fig. 40. Hermann Hankel (1839 – 1873)

Hankel as a secondary-school student:

"His achievements in this subject [mathematics – H.-J. P.] were extraordinary. During his last year at school, the headmaster gave him special permission to study the writings of ancient mathematicians in the original language instead of classical literature. This was meant to satisfy both the philological requirements of the curriculum and his enormous interest in mathematics.
This must have awoken keen interest in such a universally gifted student. Meticulously, he always strove to understand the general structure of the insights he had gained and was not satisfied until he had not found the higher principle behind these insights."[298]

Thirdly, Hankel was a pupil of Möbius'[299], who was one of the few mathematicians in Germany whose thinking was closely related to Graßmann's and to whom Graßmann himself felt a strong intellectual bond.

Fourthly, Hankel was also a pupil of Riemann's. Riemann was uniquely capable of finding the abstract and universal aspects of a problem. Combined with Riemann's profound way of thinking, which in his philosophy of nature followed Herbart, Hankel had received important insights which allowed him to understand Graßmann's thinking.

Hankel's exchange of letters with Graßmann was linked to his project of completing the *Theory of Complex Number Systems*. Hankel asked Graßmann for explanations of his approach to the theory of determinants: Hankel was planning to use Graßmann's extensive magnitudes in his book as an example for hypercomplex numbers.

After Hankel had received an answer from Graßmann, Hankel sent him a copy of his book in late 1867, explicitly asking Graßmann for a review: "In order to make my book known … it is extremely important that knowledgeable readers write reviews", he told Graßmann. "Since you have already published articles in Grunerts Archiv, I kindly ask you to review it there. I would hope that this review, despite the critical comments you might make on some aspects of the book, would nevertheless show the readers that an important branch of mathematics [has been] systematically treated there. I would be infinitely indebted to you … if you did this for me, and you would be doing a great favor to one of your admirers."[300]

Graßmann sent Grunert a review of the book, but apparently the text got lost and unfortunately it was never published.[301]

The following excerpt from one of Hankel's letters to Graßmann shows very clearly how highly Hankel valued Graßmann's opinion: "You are the only person who, in my opinion, is capable of making a competent judgment, for you are the only one to have dealt with these questions from a point of view which allows us to approach them without bias and carelessness. Your only rival is Hamilton, who is no longer among us. If my book receives your approval, I will stoically accept the arrogant ignorance and

Fig. 41. Johann August Grunert (1797 – 1872)

silence I will be confronted with, as you have been. Your approval would tell me that I have objectively contributed to scientific progress."[302]

When Hankel had finished his book, the two mathematicians ceased to communicate.

Unfortunately, Hankel's book initially did very little to popularize Graßmann's approaches: Hankel was 28 years old when his *Theory of Complex Number Systems* appeared. He was still at the beginning of his career in mathematics and did not have the kind of scientific authority that might have helped draw attention to Graßmann's *Extension Theory*. Another four years would pass before A. Clebsch used his influence to make Graßmann known in the mathematical community.

Even though Graßmann had not worked in mathematics for years, at heart he always remained a mathematician. In late 1868, for the fourth time, he hoped to become a professor of mathematics.

Königsberger had left the University of Greifswald at Easter 1869. This left a professorship of mathematics vacant. Grunert, who was one of the university's two professors of mathematics, proposed Baltzer and Graßmann for the job. But Grunert was quite isolated in his faculty and Königsberger proposed Fuchs, Dedekind and Schwarz. In a letter to the Ministry of Culture and Education, the faculty opted for L. Fuchs. Grunert was the only member to oppose this decision and added a memorandum to the letter in which he proposed Graßmann and Baltzer, but expressed a clear preference for Baltzer.[303] In the end, L. Fuchs replaced Königsberger on 20 February 1869.

In December 1868, Grunert had told Graßmann about the job opportunity in Greifswald. He assured Graßmann of his support and gave him advice on how to react and with whom to talk, should he be interested.

It is quite unclear why Grunert decided to support Graßmann. Grunert's conflicts with his faculty must have played a part in this. All of his previous assessments of *Extension Theory* had made it clear that he did not think highly of the book's scientific value. A letter to Graßmann from 1862 is just one example: "... I do not believe that your *Extension Theory* or your works on so-called extensive magnitudes make a contribution to scientific progress and I do not expect them to do so in the future. Nevertheless, they prove that you are a very perspicacious man."[304]

Graßmann had not published anything in mathematics for seven years. Graßmann's review of Hankel's book, which Graßmann had sent Grunert, had probably reminded him of Graßmann. Grunert was probably trying to make up for his strict judgments on Graßmann's mathematical achievements by giving him a slim chance of becoming a professor.[305]

Almost sixty years old, Graßmann immediately jumped at what seemed a realistic opportunity, despite the financial disadvantages. In December 1868 he told Grunert: "Should my application for the professorship of mathematics be successful, my income would be significantly reduced... But this is a sacrifice I would gladly make, for this is a position which very much suits my wishes and interests. In fact, it completely suits my whole intellectual development."[306]

To begin with, the situation at the University of Greifswald was very unfavorable to Graßmann's plans. Unfortunately, Grunert's tactical moves left Graßmann with a feeling of bitterness which would remain with him for the rest of his life.

But despite this personal disappointment, by early 1869 the scientific community was beginning to react favorably to *Extension Theory*.

Firstly, Victor Schlegel, who had been a colleague of Graßmann at the Stettin "Gymnasium" from 1866 to 1868, had begun to work on a book called *System of Geometry. According to the Principles of Graßmann's Extension Theory and an Introduction Thereto* ("System der Raumlehre. Nach den Prinzipien der Graßmannschen Ausdehnunglehre und als Einleitung in dieselbe"), the first part of which was published in autumn 1872. Victor Schlegel had spent much of his free time studying *Extension Theory* after he had taken up a position as a teacher in the small town of Waren. Schlegel's book is the first attempt by another mathematician to explain Graßmann's approach. But Schlegel was not the right propagator for Graßmann's ideas. Schlegel was incapable of creating a synthesis between Graßmann's work and the mathematical thinking of his time.

Schlegel's 1878 biography of Graßmann was much more influential than his book on *Extension Theory*. Schlegel often blindly and uncompromisingly defended his revered colleague from Stettin, thereby contributing to the cultish admiration of Graßmann in later years. But his biography raised interest in Graßmann and expressed Schlegel's sincere admiration for him.

Secondly, 1869 was also the year in which broader scientific recognition of Graßmann's achievements began to take root. In January 1869, Hankel's book drew Felix Klein's attention to Graßmann. And Graßmann's oldest son, Justus Graßmann, began to study mathematics. Surprisingly, Justus Graßmann played a part in introducing one of the most important mathematicians of the time to his father's work.

At Easter 1869, Hermann Graßmann proudly said goodbye to his oldest son, who had decided to study mathematics at the University of Göttingen. Alfred Clebsch and Moritz Abraham Stern were the leading mathematicians at the University of Göttingen. Via his son, Graßmann sent both professors a copy of *Extension Theory*, which they happily accepted. Apparently, up to this point, Clebsch and Stern had not been aware of Graßmann's writings. But this time Graßmann's work had fallen into the right hands. According to Engel[307], it was probably the silent but well-read Stern who reminded Klein twice of Graßmann's writings between 1869 and 1871, telling him: "This is all very interesting."[308]

Fig. 42. Felix Klein (1849 – 1925)

Stern had been previously interested in Graßmann's research, but Grunert did not accept him as a scientific authority. In a letter to Graßmann of 6 November 1862, Grunert remarked: "... I do not accept the algebraic analysis brought forward by Prof. Stern in Göttingen ..."[309]

Felix Klein, however, who at the time was working on his *Erlangen Program* (Klein 1974), realized that Graßmann had been a pioneer in investigating mathematical problems which were decisive for the development of modern mathematics.[310]

Felix Klein belonged to the school of mathematicians around Alfred Clebsch, and Klein urged Clebsch to study Graßmann. For his birthday on 19 January 1872, Klein gave Clebsch a device showing the linear generation of third-order curves after Graßmann's method (movable needles bound to certain constellations of intersections by a system of threads).[311]

Clebsch began to realize how important Graßmann's achievements were. He did not hesitate to introduce a wider public to the noteworthy insights he had found in Graßmann's writings. On his initiative, Graßmann was appointed a member of the Göttingen Science Society on 2 December 1871. At this meeting of the Science Society, Clebsch also held a commemorative speech for the mathematician and physicist Julius Plücker, who had died recently. F. Klein, a pupil of Plücker's, contributed to the speech, and Clebsch repeatedly mentioned Graßmann's achievements.[312]

For the first time in a quarter century, not since Graßmann received the prize for his *Geometrical Analysis* in 1846, had a renowned German mathematician publicly recognized the value of his work. And a few remarks from Clebsch were enough to arouse general interest in Graßmann. Thanks to considerable progress in mathematics, Graßmann's works could now be grasped more easily.[313] Beginning with Clebsch's followers, mathematicians began to discuss Graßmann's findings.[314]

Graßmann's heart was still in mathematics, and colleagues were finally recognizing his achievements. Clebsch's warm words and his appointment as a member of the Göttingen Science Society stimulated Graßmann to turn to mathematics once more, even though he was in the midst of publishing his *Rig-Veda* dictionary.

Fig. 43. Inauguration of a commemorative plaque for H. G. Graßmann on the wall of the Institute of Mathematics of Szczecin University on 28 May 1994, 150 years after the publication of *Extension Theory.*

Unfortunately, Clebsch died on 7 November 1872. Only ten days earlier, Graßmann had written his first mathematical treatise in ten years and had sent it to Clebsch. Clebsch could no longer propagate and elaborate Graßmann's thinking. But nothing could stop Graßmann now.

Sophus Lie was a close friend of Klein, and Klein informed him about Graßmann's treatment of the Pfaffian problem. Lie traveled to Stettin to meet Graßmann and learn about his approach to the problem, which Lie had been working on for a longer time. But, apparently, after having abandoned mathematical research for many years, Graßmann no longer felt at ease in mathematics. He had a hard time finding a useful interpretation of his classificatory approach to the problem, which at the time was groundbreaking and uniquely concise.[315]

Graßmann was becoming increasingly aware of the growing interest in his work. The obituary for Clebsch (Clebsch 1874) written by his friends also mentioned Graßmann with great admiration.[316] In 1873, V. Schlegel and Graßmann began an intense exchange of letters. Klein also began to send letters and, after Graßmann's death, he initiated the publication of Graßmann's collected works. As a sign of his admiration, the Strasbourg mathematician K. T. Reye sent Graßmann a copy of his *Geometry of Position* ("Geometrie der Lage", Reye 1877) in 1877.

Late in life, Hermann Graßmann had the satisfaction of seeing that mathematicians had finally begun to take note of his mathematical insights.

Public recognition came too late for Graßmann's career as a mathematician. Marked by illness, Graßmann continued his work in philology and dedicated whatever additional time he had to research in physics and mathematics. During the last year of his life, immobilized by serious heart problems, he wrote a large number of scientific articles in which he defended his discoveries in physics[317] and propagated older insights by linking them to the newest developments in mathematics.[318] But when Graßmann attempted to do so, his views often lacked the necessary objectivity.

Graßmann had not worked on mathematics for a long time. Relying on limited and insufficient information from libraries in Stettin, he attempted to include all new mathematical theories in his *Extension Theory*. By doing so, he was overestimating the scope of *Extension Theory*. New methods had arisen in mathematics, some of which were superior to his. There was no way his *system* could remain intact in the most recent developments of mathematics. It had to be disassembled into single approaches, ideas and methods, which could then be integrated into the living structure of contemporary mathematics. 65 year-old Graßmann was not aware of this – but it would be unjust to criticize him for this understandable inflexibility.

Graßmann was a great scientist. For a long time, nobody had understood the importance of his works. He had lived in intellectual isolation, fighting for progress in mathematics. But he had achieved great things in his life, which was now coming to an end. When he died on 26 September 1877, he was considered a great philologist, was a respected physicist and had witnessed first indications that his insights, which he had developed in small-town seclusion, were having an impact on mathematics.

His wish to become a professor of mathematics and create a school of like-minded followers had remained unfulfilled. Shortly before his death in 1877, he had the joy of learning that a second printing of his *Extension Theory* of 1844 was underway. The publisher had prematurely destroyed the unsold copies of the first printing.

Graßmann's last publication (H. Graßmann 1878) was a treatise in which he attacked materialism and atheism and called for a return to true Christian faith. Graßmann completed the foreword to this text shortly before his death. In many points it contradicted what had been his approach to scientific thinking. It was a last expression of Graßmann's diffuse path of development.

Notes

1 Quoted from Wehrmann 1906, p. 199.
2 See Wehrmann 1911.
3 Wehrmann 1911, p. 363.
4 Friedrich Engels: North- and South-German Liberalism (12 April 1842). MECW, vol. 2, p. 265.
5 See Wehrmann 1906, p. 241.
6 See in this context: Wehrmann 1894, Runze 1910, Rühl 1887, p. 18 – 23.
7 See also Wehrmann 1911, p. 404.
8 See Wehrmann 1906, p. 268.
9 Friedrich Engels: The State of Germany (8 November 1845). MECW, vol. 6, p. 23.
10 König 1972, p. 323.
11 See Wehrmann 1911, p. 451.
12 In 1822, the "Chamber" had 226 members. By 1872, the number increased to 728. – See Wehrmann 1911, p. 451.
13 See Wehrmann 1911, p. 468.
14 See Wehrmann 1911, p. 465.
15 See Wehrmann 1911, p. 464.
16 See Altenburg 1936, p. 160sqq.
17 See Wehrmann 1906, p. 270.
18 Friedrich Engels: The Constitutional Question in Germany (March – April 1847). MECW, vol. 6, p. 79.
19 Ibidem, p. 82.
20 See Wehrmann 1911, p. 476.

21 Mehring 1973, p. 105.
22 See Scherwatzky 1939, p. 201.
23 See Löwe 1870, p. 81.
24 See Wandel 1888, p. 228.
25 Ibidem, p. 156.
26 See Geschichte der Grossen National-Mutterloge 1903, p. 512sq.
27 See Wehrmann 1911, p. 490sqq.
28 See Runze 1910.
29 Ibidem.
30 Runze 1907 lists some of the school's most noteworthy pupils.
31 H. Müller 1926, p. 15. See also A. Müller 1878 and M. Wehrmann 1911, p. 489sqq.
32 See also Klein 1926, p. 173.
33 Today, this formerly Prussian town is the Polish city of Gorzów Wielkopolski. It is situated in western Poland, about 45 kilometers from the Polish-German border.
34 R. Graßmann 1876c, p. 21/22.
35 G. L. Graßmann 1868 and the Allgemeine Deutsche Biographie, vol. 9, p. 593 – 594, give a list of his other publications.
36 See G. L. Graßmann 1868 and R. Graßmann 1876c, p. 19sqq.
37 R. Graßmann 1876c, p. 26. See also: Curriculum vitae, by Justus Graßmann, reprinted in Scheibert 1937, p. 33sqq.
38 See in this context J. Graßmann's discourses for the "akademische Erinnerungsfeste" (1846).
39 See in this context and on the influence on J. Graßmann of the Romantic philosophy of nature Heuser 1996. J. Graßmann probably did not appreciate the fact that Klügel's presentation of combinatorics in Halle followed the Hindenburg school. Later he rejected this approach and relied directly on Leibniz. See KRY, p. 175.
40 See, for more details, Wehrmann 1911, p. 415sqq.
41 See, for more information, chapter 2, section 3.
42 See Runze 1910.
43 Karl Marx: Introduction to A Critique of Hegel's Philosophy of Right (1844). MECW, vol. 3, p. 177.
44 For more information on the "Turnverein" movement, see "The life of Schleiermacher" in chapter 2, section 3.
45 König 1973, p. 41.
46 See Rühl 1887, p. 56.
47 These are J. Graßmann 1817, 1824, 1835.
48 See, for more details, chapter 2, section 1.
49 Scheibert 1937, p. 36.
50 See chapter 2, section 1.

51 Scheibert 1854, p. 160/161.
52 See Scheibert 1937.
53 Becker/Hofmann 1951, p. 327/328.
54 These are, on the one hand, the "Curriculum vitae et studiorum ratio" from 17 December 1831, submitted to the Berlin commission for the first examination for teachers (25 half-folios) and, on the other, a curriculum vitae in German from 23 March 1833. Graßmann submitted the latter when he took theology exams in Stettin (15 half-folios). Both texts (Graßmann 1831, 1833) can be found in the Pomeranian Library Szczecin. Engel mentions scientific documents left behind by Graßmann (BIO, p. 373) which the author of the present book has been unable to uncover. Gert Schubring's search for additional Graßmann manuscripts was likewise unsuccessful (see Schubring 1996c). The Möbius archive in Leipzig, which possessed letters of Graßmann's, was completely destroyed during the Second World War. Therefore the present book must largely rely on the extensive documentation of Graßmann's life in (BIO). Recently discovered material on Graßmann and his family will be published in autumn 2009 in a book of Graßmann sources.
55 Graßmann 1831, translation from Latin.
56 Graßmann 1833.
57 Ibidem.
58 Ibidem.
59 Ibidem.
60 Ibidem.
61 Justus Graßmann is reputed to have said that "Hermann was more given to study than Robert, but Robert was more talented" (Scheibert 1937, p. 68).
62 H. Graßmann's secondary-school diploma, quoted from BIO, p. 17.
63 See BIO, p. 19.
64 Heinrici 1889, p. 49.
65 See, for more details, Wirzberger 1973.
66 See BIO, p. 20.
67 See the entry on Neander in Lenz 1910a, 1918.
68 See Wirzberger 1973, p. 78. For more information on Schleiermacher, see chapter 2, section 3.
69 See the entry Hengstenberg in Lenz 1910a, 1910b, 1918, and Ziegler 1921, p. 166sqq.
70 See the entry G. A. Strauß in Lenz 1910a, 1910b, 1918.
71 See Wirzberger 1973, p. 78.
72 Letter to Robert Graßmann on 26 November 1836. Quoted from BIO, p. 21.
73 See BIO, p. 21.
74 BIO, p. 24.
75 For details on the excellent standards of philological research at Berlin University, see Wirzberger 1973.

76 See the entry F. v. Raumer in Lenz 1910a, 1910b, 1918.
77 See the entry Zeune in König 1972, 1973.
78 See Wirzberger 1973, p. 78.
79 Ueberweg 1923, p. 274. – See also the entry H. Ritter in Lenz 1910a, 1910b, 1918.
80 At the time he attended Schleiermacher's classes on dialectics and on the Gospel of Matthew.
81 Graßmann 1833.
82 For more information on Schleiermacher's philosophical positions, see chapter 2, section 3.
83 Chapter 4 offers more on this.
84 For Schleiermacher's ethics, see chapter 2, section 3.
85 Graßmann 1833.
86 Quoted from BIO, p. 150.
87 Ibidem, p. 150.
88 Justus Graßmann, Hermann Graßmann's oldest son, confirmed the following information. – See BIO, p. 29.
89 See chapter 2, section 1.
90 Graßmann 1833.
91 Ibidem.
92 On 18 April 1847 Graßmann wrote the following letter to Saint-Venant, clarifying the rights to his discoveries by claiming that he had conceived the basic concepts of his *Extension Theory* as early as 1832: "Comme je lisais l'extrait de votre mémoire sur les sommes et les différences géométriques publié dans les Comptes rendus, je fus frappé par la ressemblance merveilleuse, qu'il y a entre les résultats, qui y sont communiqués et les découvertes faites par moi-même depuis l'année 1832 …". (Quoted from BIO, p. 42/43).
93 See chapters 3 and 4.
94 See BIO, p. 39.
95 BIO, p. 40.
96 Graßmann's diploma, issued on 31 December 1831; quoted from BIO, p. 41.
97 Quoted from BIO, p. 41.
98 See Engel's explanations in BIO, p. 44.
99 Biermann 1973, p. 39.
100 BIO, p. 46.
101 Report of the director of the School of Commerce K. F. Klöden to the school's administrative committee, dated 17 October 1834; quoted from BIO, p. 46.
102 Contract between the director of the trade school Klöden and J. Steiner, dated 14 October 1834; quoted from BIO, p. 48.
103 Biermann 1973, p. 38.
104 Quoted from BIO, p. 49.
105 BIO, p. 49.

106 In a letter to his father of 25 January 1835, Graßmann wrote the following on his corrections in the textbook: " …I noted a few misspellings in some formulas, which I immediately, though apprehensively, corrected …". Quoted from BIO, p. 51.
107 Letter from Hermann Graßmann to his father, 24 January 1835. Quoted from BIO, p. 53.
108 Letter from Hermann Graßmann to his father, 9 March 1835. Quoted from BIO, p. 53.
109 Ibidem, p. 54.
110 Letter from Hermann Graßmann to his brother Robert, 9 March 1835. Quoted from BIO, p. 54/55.
111 Letter from Hermann Graßmann to his brother Robert, 24 February 1836. Quoted from BIO, p. 60/61.
112 Petition to the Stettin city council by Hermann Graßmann, concerning a teaching position at the Ottoschule, dated 5 January 1836. Quoted from BIO, p. 58.
113 See BIO, p. 59.
114 Letter from Hermann Graßmann to his brother Robert, 24 February 1836. Quoted from BIO, p. 61.
115 See chapter 3, section 6.
116 For more on this, see chapter 2, section 1.
117 Letter from Möbius to H. Graßmann, 17 June 1854. Quoted from BIO, p. 66.
118 Graßmann's diploma, issued 12 July 1839. Quoted from BIO, p. 69.
119 This was the position Graßmann was hoping to obtain at the "Friedrich-Wilhelmsschule", which was to be established in October 1840 – the following year. See BIO, p. 67 (footnote).
120 Graßmann's request to the Berlin scientific examination commission, 28 February 1839. Quoted from BIO, p. 67/68.
121 Quoted from BIO, p. 69.
122 Letter of Graßmann's accompanying his examination thesis, 20 April 1840. Quoted from BIO, p. 69.
123 Graßmann's diploma, issued 1 Mai 1840. Quoted from BIO, p. 70.
124 Ibidem.
125 See BIO, p. 79.
126 Graßmann's diploma, issued 1 Mai 1840. Quoted from BIO, p. 70.
127 Graßmann's letter to the Prussian minister of culture and education Eichhorn, May 1847. Quoted from BIO, p. 124/125.
128 See: H. Graßmann 1878, p. 5.
129 This is how his brother Robert Graßmann remembered it. – See BIO, p. 73 (footnote).
130 See also chapter 4, section 1.
131 For more information, see chapter 3, section 2, and chapter 4, section 1.
132 See F. Engel in: GW11, xii.
133 For more information on Conrad (1796 – 1861), see BIO, p. 68.

134 See BIO, p. 75 (footnote).

135 The text is structured along the lines of unfolding symmetrical oppositions. The theory of language splits up into a theory of forms and a theory of concepts. Graßmann viewed the theory of concepts as a system of representations and defined it as the double opposition of sign and cognitive form on the one hand, and cognitive form and external reality, on the other. He described language as the totality of forms and ideas. At the same time, language was also the subject's construction and a representation of reality. See in this context Erika Hültenschmidt (1995), the first analysis of this text.

136 Schlegel 1878, p. 10.

137 See BIO, p. 90.

138 See BIO, p. 90 (footnote).

139 See the index of H. Graßmann's writings in BIO, p. 356.

140 See chapter 3, section 3.

141 M. Cantor and A. Leskien: "Hermann Graßmann". In: Allgemeine Deutsche Biographie 1875sqq., vol. 9, p. 596.

142 Concerning *Barycentric Calculus* (Möbius 1827), see Wußing 1984, p. 35 – 44.

143 Letter from Graßmann to Möbius, 10 October 1844. Quoted from BIO, p. 99.

144 Letter from Möbius to Graßmann, 2 February 1845. Quoted from BIO, p. 100.

145 Letter from Apelt to Möbius, 3 September 1845. Quoted from BIO, p. 101.

146 See the letter from Baltzer to Möbius, 26 October 1846. Quoted from BIO, p. 101sq.

147 Letter from Gauß to Graßmann, 14 December 1844. Quoted from GW12, p. 397/398, editor's note.

148 Letter from Grunert to Graßmann, 9 December 1844. Quoted from BIO, p. 103.

149 Letter from Gauß to Graßmann, 14 December 1844. Quoted from GW12, p. 397/398, editor's note.

150 See Stanke 1974, p. 18sq.

151 See Jaworski/Detlaf 1972, p. 413. For more information on Graßmann's theory of electrodynamics, see Sturm/Schröder/Sohncke 1879, p. 33sq. See also Müller/Pouillet 1932, p. 403, and Reif/Sommerfeld 1898sqq, p. 462.

152 Letter from Clausius to Graßmann, 15 May 1877. Quoted from BIO, p. 105. Clausius' "beloved and admired teacher" was Justus Graßmann, who had known Clausius as a pupil during his time at the Stettin "Gymnasium". – See also GW22, p. 255sqq.

153 See chapter 3, section 3.

154 Letter from Möbius to Graßmann, 9 June 1853. See BIO, p. 106. Möbius was referring to the following texts: H. Graßmann 1848a, 1851a, 1851b, 1851c, 1852.

155 Klein 1926, p. 180.

156 Klein 1926, p. 181. – See also Klein 1928, p. 132sq.

157 See Scheffer's remarks on Graßmann's theory of curves in GW21, especially on p. 383.

158 Letter from Möbius to Graßmann, 2 February 1845. Quoted from BIO, p. 109.

159 Quoted from GW11, p. 415/416, editor's note.
160 See GW11, p. 417 – 420.
161 See chapter 3, section 4.
162 Quoted from BIO, p. 111.
163 Expert opinion on Graßmann's contribution to the prize question by Möbius. Quoted from BIO, p. 112.
164 Ibidem, p. 113.
165 Ibidem, p. 114.
166 In an appendix to Graßmann's treatise (Möbius 1847), Möbius attempted to give a geometrical interpretation of these "pseudo-magnitudes". In subsequent works, Graßmann himself dropped some of these "pseudo-magnitudes". – See chapter 3, section 4.
167 Letter from Drobisch to Graßmann, 8 July 1846. Quoted from BIO, p. 115/116.
168 See chapter 2, section 2.
169 Letter from Graßmann to the Prussian minister of culture and education Eichhorn, Mai 1847. Quoted from BIO, p. 125.
170 Ibidem, p. 125.
171 "Written assessment by Prof. Kummer on the mathematical writings of secondary-school teacher Hermann Graßmann from Stettin and his intentions of carrying out academic research", 12 June 1847. Quoted from BIO, p. 126.
172 Ibidem, p. 126.
173 Ibidem, p. 127.
174 Ibidem.
175 Ibidem.
176 Communiqué from the Stettin school administration to the ministry of culture and education, 31 July 1847. Quoted from BIO, p. 130.
177 Letter from Eichhorn to H. Graßmann, 4 September 1847. Quoted from BIO, p. 130.
178 Wehrmann 1911, p. 471/472.
179 See Wehrmann 1911, p. 470sqq.
180 Letter from Graßmann to Möbius, 22 May 1853. Quoted from BIO, p. 160.
181 See Dahlmann 1835, p. 80sq.
182 Dahlmann 1835, p. 179/180. Concerning his rejection of democracy and despotism, see Dahlmann 1835, p. 13sqq.
183 Karl Marx: The Bourgeoisie and the Counter-Revolution (December 1848). MECW, vol. 8, p. 162.
184 See chapter 1, section 1.
185 Friedrich Engels: Marx and the Neue Rheinische Zeitung (1848 – 49). MECW, vol 26, p. 123.
186 See Altenburg 1936, p. 160sqq.

187 For more information, see Bartholdy 1907, Heintze 1907, Ilberg 1885, Runze 1907, Runze 1910, Schulze 1939.
188 Droysen 1911, p. 34sq.
189 Schulze 1906, p. 40.
190 Friedrich Engels: Revolution and Counter-Revolution in Germany (1851/52). MECW, vol. 11, p. 34.
191 Letter from Graßmann to Möbius, 22 Mai 1853. Quoted from BIO, p. 160.
192 H. Graßmann, "Die Früchte des Berliner Barrikadenkampfes". In: Königlich privilegierte Stettinische Zeitung, 15 April 1848. Quoted from BIO, p. 138/140.
193 R. Graßmann 1890a, p. xxi/xxii (footnote).
194 Deutsche Wochenschrift 1848, number 5, supplement.
195 See Deutsche Wochenschrift 1848, p. 23/1.
196 Ibidem, p. 2/1.
197 Ibidem, p. 23/1.
198 Ibidem, p. 2/1.
199 See ibidem, p. 2/2 and number 5, supplement.
200 Ibidem, p. 2/2.
201 Ibidem, p. 23/2.
202 Ibidem, p. 2/2.
203 Ibidem, p. 10/1.
204 Ibidem, p. 17/1.
205 Ibidem, p. 18/1.
206 Ibidem, p. 21/2, 22/1.
207 Ibidem, p. 28/2.
208 Ibidem, p. 27/1.
209 Ibidem, p. 29/1.
210 Ibidem, p. 30/2.
211 Ibidem, p. 30/2.
212 See Friedrich Engels: The Constitutional Question in Germany (1847). MECW, vol. 6, p. 78.
213 Deutsche Wochenschrift 1848, p. 23/2.
214 Marx/Engels: The German Ideology (1845/46). MECW, vol. 5, p. 195.
215 Heine 2007, p. 114.
216 Ibidem.
217 For more information on Schleiermacher's *Dialectic*, see chapter 2, section 3.
218 Karl Marx: Introduction to A Contribution to the Critique of Hegel's Philosophy of Right (1844). MECW, vol. 3, p. 185.
219 See R. Graßmann1890a, p. xxi, footnote.
220 Letter from Graßmann to Möbius, 22 May 1853. Quoted from BIO, p. 160/161.

221 See Scheibert 1937, p. 45 – 50.
222 Ibidem, p. 49.
223 BIO, p. 248.
224 Letter from Graßmann to Möbius, 22 May 1853. Quoted from BIO, p. 161.
225 See the letter from Möbius to Graßmann, 9 June 1853. Extracts in: BIO, p. 162/163.
226 Bellavitis believed to have shown that Graßmann's methods for generating third-order curves (H. Graßmann 1848a) were not generally valid. But Graßmann gave the unconditional proof of their generality and, at the same time, uncovered Bellavitis' mistakes (see H. Graßmann 1855g). Bellavitis only learned about Graßmann's article in August of 1859. He immediately reacted publicly to his mistake and informed Graßmann in a letter on 26 March 1860. – See BIO, p. 106.
227 According to Engel, *Arithmetic* and the second *Extension Theory* (A2) were published in 1860 and 1861. For editorial reasons, the dates were changed to 1861 and 1862. – See BIO, p. 225 and 230.
228 See chapter 1, section 4.
229 Knoblauch had proposed Graßmann for full membership in the Society for Natural Science Research ("Naturforschende Gesellschaft") in the town of Halle. Graßmann was made a member on 6 February 1864. – See BIO, p. 267, footnote.
230 See Helmholtz 1855.
231 See, for more details, Turner 1995.
232 H. Graßmann 1853, p. 161.
233 Ibidem, p. 162.
234 Ibidem, p. 163.
235 Ibidem, p. 168.
236 Ibidem, p. 171.
237 See also Wußing 1977, p. 41sqq.
238 Friesler 1953, p. 93.
239 Letter from Graßmann to Möbius, 22 May 1853. Quoted from BIO, p. 162.
240 Concerning the importance of Graßmann's approach for Helmholtz' theory of sensation, see Lenoir 2004.
241 Friesler 1953, p. 105.
242 See Rodnyj/Solowjew 1977, p. 163.
243 See Trendelenburg 1961, p. 179sqq.
244 H. Graßmann 1877d, p. 231.
245 See Trendelenburg 1961.
246 See the article by M. Cantor and A. Leskien in Allgemeine Deutsche Biographie, vol. 9, 1875sqq, p. 597.
247 See Junghans 1978, p. 251sq.

248 BIO, p. 244/245. This section also offers more information on Graßmann's law of comparative philology.
249 See also Elfering 1995.
250 See Junghans 1978, p. 252. See also Reich 1995.
251 See BIO, p. 154sq.
252 See BIO, p. 159, Wandel 1888, p. 241sq.
253 See BIO, p. 209sqq.
254 Müller 1909, p. 344.
255 Wandel 1888, p. 254.
256 Quoted from BIO, p. 262.
257 Quoted from BIO, p. 265.
258 Quoted from BIO, p. 265.
259 G. Schubring (1991) has written a very noteworthy introduction to what it meant to be a teacher of mathematics in the 19th century.
260 Quoted from Schubring 1991, p. 169.
261 Ibidem.
262 For more information on Graßmann as a teacher and on the Stettin "Gymnasium", see BIO, p. 255sqq, Bartholdy 1907, Delbrück 1877, Heintze 1907, Müller 1909, Runze 1907, Runze 1910, Schlegel 1878, Wandel 1888, Wehrmann 1894. See also the summary in Schwartze 1996.
263 Quoted from BIO, p. 161.
264 See the selected texts from the correspondence between Graßmann and Möbius in BIO.
265 Möbius was referring to the following articles:
"Sur les clefs algébriques". In: Comptes Rendus 36 (1853), p. 70 – 75 and p. 129 – 136.
"Sur les avantages que présente, dans un grand nombre de questions, l'emploi des clefs algébriques". In: Comptes Rendus 36 (1853), p. 161 – 169.
And Möbius was also thinking of Saint-Venant's text: "De l'interprétation (géométrique) des clefs algébriques et des déterminants". In: Comptes Rendus 36 (1853), p. 582sqq.
266 Letter from Möbius to Graßmann, 2 September 1853. Quoted from BIO, p. 175.
267 Letter from Graßmann to Möbius, 19 February 1854. Quoted from BIO, p. 182.
268 See the letter from Graßmann to Möbius, 7 January 1855. The section in question has been reprinted in BIO, p. 192.
269 The following lines on Cauchy and Saint-Venant rely on Engel's work in BIO, p. 120 – 122 and p. 195 – 201.
270 BIO, p. 203.
271 Graves 1889, p. 441sq.
272 Letter from Hamilton to De Morgan, 2 February 1853. In: Ibidem.
273 Quoted from Bell 1986, p. 360/361.
274 See chapter 2, section 2.

275 See BIO, p. 202sq.
276 LA, p. v.
277 See LA, p. 1.
278 See chapter 3, section 6.
279 See the letter from J. A. Grunert to H. Graßmann, 9 December 1844. In: BIO, p. 103.
280 Note from Graßmann to minister of culture and education von Bethmann-Hollweg, 14 January 1862. Quoted from BIO, p. 232.
281 Letter from minister of culture and education von Bethmann-Hollweg, 14 January 1862. Quoted from BIO, p. 232.
282 Letter from H. Graßmann to H. Hankel, 8 December 1866. Quoted from BIO, p. 272.
283 Ibidem.
284 See BIO, p. 302sqq.
285 Müller 1878, p. 346.
286 BIO, p. 254.
287 In Robert Graßmann's recollection. – See BIO, p. 303.
288 See BIO, p. 304.
289 See Wandel 1888, p. 250, and Reich 1995.
290 See Toporov 1960, p. 235.
291 H. Graßmann 1870, p. 1.
292 Ibidem.
293 Ibidem, p. iii.
294 See Bezzenberger 1873.
295 Letter from Graßmann to Hankel, 8 December 1866. Quoted from BIO, p. 271.
296 Letter from Hankel to Graßmann, 24 November 1866. Quoted from BIO, p. 270.
297 See Bell 1986, 340sqq.
298 Zahn 1874, p. 583/584.
299 See Zahn 1874, p. 584.
300 Letter from Hankel to Graßmann, 4 June 1867. Quoted from BIO, p. 276.
301 See BIO, p. 278.
302 Letter from Hankel to Graßmann, 4 June 1867. Quoted from BIO, p. 276.
303 See BIO, p. 285/286.
304 Letter from Grunert to Graßmann, 1 June 1862. Quoted from BIO, p. 242.
305 In the context of Robert Graßmann's examinations as a teacher in 1839/40, G. Schubring has pointed to the fact that Grunert was friendly with the Graßmann family. See Schubring 1996d, p. 65.
306 Letter from Graßmann to Grunert, January 1869. Quoted from BIO, p. 280.
307 See BIO, p. 312.
308 BIO, p. 312.
309 Letter from Grunert to Graßmann, 1 June 1862. Quoted from BIO, p. 244.

310 See chapter 3, section 7.

311 See BIO, p. 312, footnote.

312 See Clebsch 1871, p. 8 and p. 28.

313 See chapter 3, section 7.

314 For more information on the reception of Graßmann by Clebsch and his school, see Tobies 1995.

315 See BIO, p. 318sq.

316 See Clebsch 1874, p. 12, 31 (footnote).

317 See Graßmann 1877b, 1877d.

318 See Graßmann 1877e.

2 Graßmann's sources of inspiration

2.1 Justus Graßmann: Father and precursor of his son's mathematical and philosophical views

As we have already learned in previous sections, Hermann Graßmann and his father shared a common approach to scientific research. Unfortunately, this relationship has received little attention in scientific literature. Apart from the author's dissertation from 1978 (Petsche 1979a), on which we will largely rely here, recent publications have discussed the influence of the Romantic philosophy of nature on Justus Graßmann (Heuser 1996) and the influence of Justus Graßmann's vector algebraic approaches on the work of Hermann Graßmann (Scholz 1996). An analysis of Justus Graßmann's ideas concerning the structure of mathematics and the foundations of arithmetic, and of the subsequent impact of these ideas on his son, has also been published (Radu 2000).

All of these scientific texts show that Hermann Graßmann's creativity and its specific characteristics can only be fully understood when Justus Graßmann's theoretical positions are taken into account as well. Only by analyzing his father's ideas may we reach deeper insights concerning the many points of contact between father and son. We will also have to analyze the intellectual micro-climate in which Hermann Graßmann's scientific positions developed. We will begin with a short overview of Justus Graßmann's scientific achievements and pay special attention to the elements relevant to the work of Hermann Graßmann.

The mathematical and philosophical positions of J. Graßmann developed gradually. His textbooks for a basic education in mathematics of 1817 and 1824 (J. Graßmann 1817, 1824) were a first step, followed by his 1827 treatise *On the Concept and Extent of the Pure Theory of Number* (ZL), the publication of his work on the geometric theory of combinations (KRY) in 1829 and the textbook of trigonometry of 1835.[1]

Inspired by Joseph Schmid's *Lessons on Forms* ("Formenlehre", 1809), J. Graßmann attempted to achieve an elementary synthesis of combinatorics and geometry in 1817 and 1824. In the process, he discovered an elegant way of approaching simple crystalline structures mathematically. Taking thoughts from Kant and Leibniz as a point of departure, the new approach to geometry inspired him to undertake a philosophical analysis of the object of mathematics and to develop his ideas on a "geometrical theory of combinations", investigating its usefulness in the context of crystallography.

Point for point, we get the following overall picture:

The establishment of a number of elementary schools for the poor, for which J. Graßmann and his friend, school councilor Bartholdy (himself a close friend of Schleiermacher[2]), had volunteered to provide teaching material (for teachers "lacking a truly scientific education"[3] and their pupils), provided the occasion for the publication of two textbooks for basic education in mathematics.[4]

From J. Graßmann's *Geometry* (1817):
"General preliminaries. How to focus on the teacher's words and acts by speaking in chorus correctly."

"The children have assembled calmly in front of, or around the teacher. The teacher and the pupils can see one another well: the children's hands are on the table, side by side, or loosely hanging by their hips.

Teacher. (from now on **T**) Children, pay attention to what I do and say! (while lifting his right hand) 'I am lifting my right hand.'

T. Now do as I do. But wait until I tell you to and give you the sign.

– Lift your right hand! (The teacher gives the sign by lifting his hand slightly, without lifting his arm. This we will designate by '**S!**'). There should be a short pause between the teacher's telling the children what to do and his giving the sign, for the children will have to think about what they are supposed to do. (This short pause will be designated by '**P!**': therefore the whole paragraph could be reduced):

T. Right hand up! – **P.** – **S!**
The children do as told. (from now on '**Ch. d.**')

If a child should lift the hand before the teacher has given the sign, or if it lifts the left hand instead of the right, the mistake will be corrected lovingly.

T. Pay attention and do what I tell you to! – Right hand back! – **P.** – **S.** – **Ch. d.**"[5]

Next, we practice a rhythmical chorus, conducted by the movements of the teacher's hands. Then the children learn to answer the teacher in complete sentences: "From now on, when you answer me, tell me every time what you are referring to and what you have to say about it!"[6]

"Only now the children have gained full awareness for everything they are doing..."[7]

Influenced by the pedagogical school of Pestalozzi, especially after the War of Liberation of 1813/14, Justus Graßmann's views on the teaching of mathematics took shape. He finally expressed them in his *Geometry for Elementary Schools* ("Raumlehre für Volksschulen", J. Graßmann 1817, 1824). This textbook, which was reviewed favorably by the progressive pedagogical theorist Diesterweg[8], took up ideas from a similar textbook, the *Lessons on Forms* (Schmid 1809) by Pestalozzi's collaborator Joseph Schmid. In the introduction to his book, J. Graßmann explicitly referred to basic ideas from Schmid's book, despite his reservations concerning Schmid's practical way of applying these principles.

Joseph Schmid submitted traditional Euclidean geometry to a critique, questioning its pedagogical and philosophical value.[9] Schmid rejected the Euclidean way of representing and teaching geometry, which – according to him – was "*worthless* as a manner of educating human nature" because it lacked an "internal *organic* connection to the way in which human capabilities unfold [emphasis mine – H.-J. P.]". According to Schmid, the development of human capabilities did not "parallel it"[10]. Instead, Schmid claimed, Euclidean geometry tended to lose its explanatory value by "working on isolated and fragmentary geometrical problems"[11].

Therefore, J. Schmid arranged Euclidean problems in a pattern. This pattern was structured along the lines of form (constructions using the compass and/or the straightedge) and magnitude[12]. J. Schmid developed his geometry following the ideas of a mathematician he was "friendly with": "One should begin geometry by dealing with everything that has to do with one straight line, then, with two, etc."[13]. Therefore, the theorem one was using should not seem to be an "arbitrary concoction of problems". Instead, "the problems … should necessarily follow from the idea which formed the point of departure"[14]. From this combinatorial and synthetic perspective, geometry, which to him was a "formal theory of intuition"[15] in Pestalozzi's sense, therefore did *not* begin with definitions. Definitions had to appear in the conclusion – "for how could a form [be grounded in] an intellectual concept … before … we have even seen it"[16].

We can find J. Schmid's views in the basic principles on which J. Graßmann relied in writing his textbooks on geometry[17]:

1 Geometry begins with *intuition*, not concepts, and it therefore must also be based on intuition. That is to say that, primarily, geometry is an act of *construction* in *pure* intuition. The subsequent conceptual analysis is built on this first step.

2 Geometry consists of two elements: Linear and angular-magnitudes (lines and angles). Geometry has a larger number of elements than arithmetic, which only begins with one. Methodologically, this means that one must begin with the first element, then go on with the second, and finally analyze the relationship between the two. Therefore, according to J. Graßmann, one can easily understand and arrange truths, which is not the case in the Euclidean approach.

3 The material is organized in analogy with the structure of an *organism*. Therefore, at first, we attain smaller, connected *entities*, which are the *components of a larger whole*. The representation of this structure, J. Graßmann said, is a kind of *artwork*.

J. Graßmann began his reflections by generating geometrical objects through *movements*[18] and by considering the point the "limit of all extension"[19]. He investigated lines "in relation to the number and position of ... ensuing points of intersection and angles"[20]. This approach of incidence-geometry was, essentially, a combinatorial approach to geometry. The second part of *Geometry* was an "application of conjunctions from the general theory of magnitudes to spatial objects"[21].

This "kind of intuitive geometry with no strict mathematical foundation"[22] (Moritz Cantor) developed, step by step, the "addition of lines", the "subtraction of lines", the "multiplication of lines"[23] etc. and conceptualized its theories in a peculiar German terminology: for example, to multiply = "veröften", the multiplicand = "Oeftstoff", product = "Geöft", factors = "Veröfter", dividend = "Theilstoff", divisor = "Theiler", quotient = "Theilfund", etc.

We should take note of three essential aspects of J. Graßmann's fundamental principles in his *Geometry for Elementary Schools*:

1 The first principle, which arose from Pestalozzi's theories[24], shares fundamental traits with the Kantian concept of mathematics, especially of geometry[25]. The procedure of constructing geometrical objects from "pure intuition", which then gives rise to geometrical truths, is completely in the Kantian tradition. But, just as in Pestalozzi, J. Graßmann did assume that "pure intuition" was the whole picture: to him, the constructions of the world (as a kind of organism) remained in place, side by side with the reconstructive efforts of the human mind.

2 The second principle led to a combinatorial approach to geometry. It inspired J. Graßmann to analyze the fundamental principles of the theory of combinations. But this brought him under the sway of the scientific and theoretical program begun by Leibniz, who – as one of the creators of the theory of combinations – had applied it methodically to all fields of scientific thought[26], submitting its foundations to a logical and philosophical analysis.

In a later treatise, *On Physical Crystallonomy and the Geometrical Theory of Combinations* ("Zur physischen Krystallonomie und geometrischen Combinationslehre"), Justus Graßmann explicitly emphasized the fact that his approach to the theory of combinations had been inspired by Leibniz and that he disliked the clumsy mode of presentation of the Hindenburg school.[27]

3 The third principle brings forward the view that science is an "organism", an "artwork", which identifies J. Graßmann as a follower of the German Romantic philosophy of nature (Schelling, Steffens, Schleiermacher, and others). This philosophy was especially important for the dialectical parts of his view of mathematics.

J. Graßmann believed in many of the essential elements of bourgeois German national education and participated actively in the German patriotic movement. Therefore, further reflections on the object and essence of mathematics led him to classical bourgeois philosophy and pedagogy, especially Leibniz, Schelling, Pestalozzi and Schleiermacher. Joseph Schmid also made decisive contributions here. He wrote: "When a mathematician is a great mathematician, but merely a mathematician ..." this mathematician does not participate "in the freedom, liveliness and independence of all spiritual life in nature and human existence"[28]. "The well-educated mathematician will ... necessarily have to be aware of himself, and of the external and internal nature of his surroundings; he will become a physicist, a philosopher."[29]

The reconstruction of mathematics for elementary schools and the establishment of new elements, obtained through sense-perception and synthesized in inner intuition to form a stringent mathematical structure, created new principles for the foundation of mathematics. These principles would probably never have arisen from purely mathematical problems. Therefore, in the introduction to the *Geometry* of 1817, J. Graßmann wrote:

"This is not the place to present my views on mathematics, and I will have to do this some other time. Nevertheless, I would like to note that, just like the synthesis of

Leibniz on the "combinatorial method":

"There are two methods, the synthetic, aided by the science of combinations, and the analytical. Both methods are capable of showing the origins of constructions; this is not exclusive to the analytical method. The difference between the two lies in the fact that combinatorics, by taking the simpler elements as its point of departure, is capable of forming an entire science, or at least a series of theorems and problems, among them, the problem one was trying to solve. Analysis, in contrast, takes the problem back to a simpler level.
...The most complete scientific method, though, will not begin with what naturally comes later, with combinations and the particular, available for sense-perception, but with the simplest and most general concepts and truths, which show us first from where they gradually descend towards the particular and towards combined concepts. It obeys the laws of synthesis or of the science of combinations, which shows how the different species arise systematically from the highest, mixed classes... Once we possess it, the synthetic method will be the clearest and the simplest we can possibly have."[30]

sameness generates the magnitude, the conjunction of different elements generates the combination, this difference may manifest itself in the distinctness of the elements, or in their succession in time and space. Therefore, geometry also possesses its theory of combinations, and the present booklet is an attempt at presenting its points of departure."[31]

This already hints at Justus Graßmann's dialectical schema, which he developed later on and which Hermann and Robert Graßmann took up in their project of finding a new foundation for mathematics. In 1827, Justus Graßmann published a text called *On the Concept and Extent of the Pure Theory of Number*[32] ("Über den Begriff und Umfang der reinen Zahlenlehre", ZL). It was yet another expression of his intense interest in the foundations of mathematics and in the search for their philosophical essence. Its foundation of the theory of number from the perspective of a philosophy of mathematics went far beyond what Kant brought forward during his "critical" phase. It would prove to have a decisive influence on Hermann Graßmann's views.

In the book, Justus Graßmann assumed that the increase of knowledge in mathematics raised the risk that its presentations might become "blind instruments that one can only apply, as it were coincidentally, to the appearances of nature, rather than their attending the explanation of nature in an orderly way, or rather, since they are independent of it, their being a model and standard for its presentation."[33] J. Graßmann had realized that it was necessary to develop a coherent and applicable theory for the investigation of the phenomena of nature. This realization – tightly linked to his religious beliefs – prompted him to look for structures in raw information, "thus to elevate the raw information to a genuinely ordered knowledge".[34] This was to say that "the function of every part is distinctly perceived relative to the whole, and so that this latter can appear as an *organism*, as manifestation of an infinite intellect [emphasis mine – H.-J. P.]."[35]

Über
den Begriff und den Umfang
der reinen Zahlenlehre.

Je mehr die Wissenschaften an Umfang gewinnen, desto nothwendiger wird es, die sich darbietenden Massen zu gliedern, nicht bloß um dem Anfänger den Eintritt zu erleichtern, sondern, was die Hauptsache ist, um dadurch das rohe Wissen zu einem geordneten wahrhaften Wissen zu erheben, welches die Stellung, den Zusammenhang, die Funktion jedes Theiles in dem Ganzen deutlich erkennen, und so dieses letztere als ein organisches, als eine Offenbarung eines unendlichen Geistes, wie sie uns in einer bestimmten Sphäre klar geworden, erscheinen läßt.

Fig. 44. The first lines of *Pure Theory of Number*

This task, which showed all the signs of Romantic philosophy and its concept of nature, apart from strong religious overtones, made it necessary to explicate the "elements of science" and present "the inner connections of the operations and constructions, allowing each step occasioned by the nature of the subject to enter as such, and, granting the intellect a resting point in the results achieved, simultaneously use them as a starting point for higher developments."[36]

The Romantic philosophy of nature in Germany:

"The most important contribution of the so-called Romantic philosophy of nature … consisted in its confidence in science, which was to serve as a tool for gaining certainty about nature and mankind. The laws of reason and the phenomena of nature served as a model for human life. An Enlightenment utopia, this philosophy ignored the Hegelian oppositions and promised to create a harmony between worldview *and* life. Placing its trust in science and in the potential of science for providing a kind of harmony, this movement favoring the study of nature compensated for the difficult German reality and the scientific and technological underdevelopment which plagued Germany…"[37]

J. Graßmann then went on to offer a "constructive" and genetic foundation of mathematics, which brought him into the intellectual vicinity of Kant[38], by determining the essence of mathematics as the process of generating "its first concepts by a synthesis characteristic of it (which we call a construction in the broader sense)"[39]. But J. Graßmann drew a line between himself and Kant when he rejected the connection between mathematical synthesis and "synthetic judgments" in the Kantian sense. He gave an example by demonstrating the synthesis of the number (1 + 1): " …but nothing can be said here about the objective validity of this synthesis – that is, whether one unit really amounts to another –, the conjunction can take place unconditionally, and the concept created thereby, the product of this synthesis, is the number two."[40]

We could very well call this view of the content of mathematical theorems a modern one: "Now mathematics establishes theorems in which a subject is conjoined to a predicate, not in an arbitrary way, to be sure, but rather with respect to their content; but this content is precisely only such as is laid down by the synthesis as equal or unequal. Thus these theorems are only expressions of the nature of that characteristic mathematical synthesis, and that which is given along with it."[41] Because of his intense interest in the theory of combinations – J. Graßmann had already begun to work on his treatise of 1829 (KRY)[42] – he reached a generalized understanding of the object of mathematics which went beyond the concept of magnitude. Differing from logic, the object of mathematics was something that was posited without any prior assumptions about its content. With the fundamental synthesis in mind, the object then had to be posited as "identical" or "different". But J. Graßmann maintained: "No equality can be imagined without inequality. … Equality and inequality are only factors with respect to those to be conjoined, and it is only a question of which of the two applies in the fundamental definition of the synthesis, although both are always at hand."[43]

Later, Hermann Graßmann took up and extended his father's dialectical developments, which Justus Graßmann had used to create the principles for structuring the mathematical disciplines.[44]

Justus Graßmann's analysis of the foundations of the theory of number, which followed these general lines of inquiry and began by discussing the "*origin* of the number concept", giving its "genetic interpretation"[45], is relevant from a philosophical point of view. According to J. Graßmann, only the "opposition between the one and the many"[46] could create the concept of number. "The concept of a number therefore comes to the point, that this specific multiplicity [that is, a plurality of things in thought – H.-J. P.] is conjoined in a unit of consciousness. – Each of the conceptions imagined as similar one calls the *unit* in relation to number."[47]

But, concerning concepts, J. Graßmann made a clear distinction between explaining why they arose, and their scientific definition. "The preceding interpretations", he wrote, "are genetic, i. e. they establish the possession of the concept and incidental reasons for its origin. However, number as such must be regarded as the specific *quantity of the setting of the unit*."[48]

His concept of concrete and abstract numbers is remarkable in this context, because it has lost nothing of its relevance today. "Now a number", he remarked, "is called concrete or 'named' if the unit is a specific conception; if however it is only the conception of a general given, without specific content (the conception of a conception ["die Vorstellung einer Vorstellung" – H.-J. P.]), then the number is called abstract or unnamed."[49]

Therefore, in Graßmann's view, the pure theory of number had to be disconnected from all objects in experience, which represent the content of *specific* conceptions.

Opposing the view – which we can also find in Kant – in which the unit is conceived as a "continuous magnitude"[50], J. Graßmann explained that pure arithmetic, even if this were the case, had to be an abstraction. He also argued that, after all, indivisible units existed in nature and in thought, such as three persons or three concepts. This, he believed, justified his view.

In this context, J. Graßmann made a sophisticated argument in favor of atomism in physics, interpreting this as an *attempt* to subject "nature to whole numbers, thus to encompass its evolutions within the relations of the theory of number and so understand them."[51] In today's terminology, this amounts to saying that the model of natural numbers serves to express certain elements of the regularities in the laws of nature.

J. Graßmann conceptualized the arithmetical operations with natural numbers, addition, multiplication and exponentiation, and their inverse operations, along the lines of a single principle of development as three orders of enumerating or resolving: "The intellectual activity of the production of a number from the unit one calls *enumeration*, and this consists in the comprehension of given units in a unit of consciousness. … Enumeration and decomposition are the two fundamental intellectual activities in numerical arithmetic; all arithmetic construction must ultimately be reduced to them."[52]

To J. Graßmann, this conception meant that multiplication was the *enumerating of equal numbers* (equal numbers are the new units). "The product of this enumeration is a number of numbers, but of specific ones, e.g. 4 nines, the so-called multiplicative conjunction."[53]

Furthermore, he conceptualized a power – in accord with his guiding idea – as the *enumerating of equal factors* (factors, again, are the new units), for example 4 * 4 * 4 * 1. The *exponent* is the "third-order" number resulting from this conjunction, the unit which is enumerated the *root*.[54]

For J. Graßmann, there were no further ways of enumeration. The conjunctions mentioned so far were synthetic conjunctions, the inverse operations, the decompositions, *analytical*.[55]

P. Lorenzen on the operative foundation of mathematics

"For 'enumerating' is the meaning of numbers – and enumerating is an important part of those actions which lead the mathematician to make statements on schematic operations."[56]

Finally, we will have to focus on Graßmann's remarks on the possibility of interchanging elements of conjunctions in arithmetic: "In addition", he wrote, "the nature of the constituents of a composition are not distinguished at all, in multiplication they are distinguished by the concept, but as numerical values they are mutually indifferent; for exponentiation however the distinction is fully developed, root and exponent are distinguished, not only as concepts, but in addition one can no longer interchange them."[57]

He even went a step further in his *Trigonometry* (1835). Here he explained that not just the interchangeability of the constituents of a conjunction presented a problem, but that the product itself had a different quality to it than the constituents of the synthesis. Not only did this mean that negative numbers could not be part of a *pure* theory of number, but also that, strictly speaking, the multiplication of numbers did not simply give a number, but "a number of a higher order"[58]. Therefore, while the pure theory of number left no room for negative numbers, and the multiplication of numbers still gave numbers, for the first time, the subtraction of displacements in geometry had become possible.[59] The product (of points/lines) led to new elements (lines/bodies).

Justus Graßmann created a framework for the pure theory of number by presenting a system of different orders of arithmetical conjunctions: pointing to his interest in the philosophy of nature, he had wished to call them[60] the mechanical (addition), the chemical (multiplication), the organic (an exponentiation), but he did not dare to do so (and therefore he designated them as mechanical, chemical and *dynamic* conjunctions). The field of the pure theory of number "[i]n its length ... is defined by the three orders of enumeration, in its breadth by the pure unit, as the notion of a simple given. ... Its depth is unlimited, as every organic entity, which is indeed externally defined, and assumes an easily graspable form, but conceals within an infinite depth, whereby it is a subject of never-ending investigation."[61]

J. Graßmann considered this *internal* perspective on the theory of number to be, generally speaking, a desirable ideal. In a way, it had to represent "the prototype of pure scientific thought".

"But every interior understanding is simultaneously an understanding with nature, whose law of exterior development cannot be more manifest to us than we have developed the law of the interior synthesis within ourselves"[62]. Graßmann wanted to underline the view that the human mind played an active role in the process of recognition. He also insisted on the importance of theological and constructive ways of thinking when it came to understanding empirical reality.

Summing up, we can say that Justus Graßmann, inspired by the Romantic philosophy of nature and critical of Kant, managed to create an independent, truly philosophical analysis of the foundations of mathematics with his *Pure Theory of Number*. The Romantic philosophy of nature had taught him to consider the sciences from an all-encompassing perspective, and this perspective led him, while he was trying to find a logical solution to conceptual and dialectical contradictions, to a deeper understanding of mathematics. His views were imbued with the optimistic attitude the Romantic philosophy of nature maintained towards scientific work.

In his *Pure Theory of Number*, J. Graßmann repeatedly pointed to the structural similarities which, as he saw it, existed between the theory of combinations and arithmetic. Thanks to his analysis of the foundations of mathematics, he now possessed the heuristic tools he needed to develop his geometric theory of combinations.

A reader of the 1829 treatise *On Physical Crystallonomy and the Geometrical Theory of Combinations. First booklet.* will feel that J. Graßmann truly considered himself the founder of a new mathematical discipline. "The sheer quantity of treatises and their topics", he wrote in his introduction, " – which confuse us as they come to us every day – makes it easy to overlook things, even when they are truly innovative and useful, as long as they do not catch our attention with insolence and shrewdness. Here, we are dealing with the presence and pertinence of a new, independent mathematical science, which aims to create simple and complex structures, to show their interrelationships and which claims to grasp all crystalline structures in one single approach."[63]

Even though J. Graßmann massively overestimated the weight of his achievements when he declared himself the founder of a new science[64], his methodological technique was exemplary and served as a model for his son. Instead of obscuring the path towards his theoretical developments, he explicitly aimed to make his points of departure as clear as possible. "While I am patiently waiting … to hear from the specialists", he wrote, "I nevertheless feel obliged to give a brief explanation of how my approach to this subject *arose* and *slowly developed*. I will also say some things about the *content*, the *scope* and the *form* of the present and the following booklets, especially since the process of their development might be important to readers trying to evaluate my work [emphasis mine – H.-J. P.]."[65]

He explained how his research on the theory of combinations, which was not built on the concept of magnitude, showed him that the old concept of mathematics was too limited and how this insight inspired him to redefine the essence of mathematics, on the one hand, and to reassess the theory of combinations, on the other.

Referring to ideas from *Geometry* (1817) and *Pure Theory of Number* (1827), J. Graßmann went on to say that the theory of combinations was founded upon the "differences of the given", which created "a direct contrast with arithmetic, and … completely disconnects the deepest essence [of the theory of combinations] from arithmetic. Therefore it becomes a particular, completely independent science, which nevertheless maintains a certain parallelism with arithmetic and permits similar syntheses."[66]

At a time when mathematics was evolving rapidly, he nevertheless emphasized the need for basic research in this field. "The mathematician should not just increase the size of his scientific field by increasing the complexity of his theories, while he neglects the traditional inventory of his science or merely passes through it on the old beaten paths. Instead, he should investigate it from a new perspective, classifying and extending it …"[67] The achievements of Hermann Graßmann made it clear how important this demand was.

J. Graßmann vigorously defended the "ideal of purity" in mathematics. But this position was not a consequence of a lack of appreciation for empirical research. For only when the result of mathematical development "is a perfect representation of the synthesis of the spirit", J. Graßmann wrote, "[will it] approach the synthesis of nature and its manifestations all by itself, … just as, inversely, the deeper understanding of nature can give the mathematician a hint about where he must still improve, innovate and construct his system."[68]

His motivation was religious in nature. To J. Graßmann, the conscious representation of the "concordance of nature and spirit" was the "process of liberating the human mind from the torture of empty abstractions and raw empirics."[69] His epistemological optimism was based on the belief that God had created man and the external world in His image, following a coherent plan. All "the external appearances of nature can and must be the manifestation of something hidden inside. They must be approached like a hieroglyph which nature has given us to read and which confronts us with the task of deciphering"[70]. This is how J. Graßmann put it in his *Crystallonomy* (1829), and he went on to say: "Mathematics can only regain the position it deserves, it can only unveil its secret depths if we understand that it is the internal complement to the external laws of the world, even though we may never grasp the latter completely."[71]

His claim that philosophy, the natural sciences and mathematics had to join forces was groundbreaking and remains so today, apart from inspiring his son: "May an enlightening and healthy philosophy, committed to clarity and not fearful of intellectual depth, build a bridge between speculation and observation, showing why they both deserve our appreciation …".[72]

By explaining the genesis of his approach to a geometrical theory of combinations, J. Graßmann uncovered first elements of a way of treating geometry without metrics. It had become possible to conceptualize spatial objects as combinations, if one left "all ideas concerning proportion of magnitude"[73] aside. "We have a method", he wrote, "to create relations between lines and surfaces in such a way that, automatically, a large number of simple and regular objects develop from them …"[74] This, J. Graßmann announced, was the method he was presenting in his book. The determination of "lines by their position and direction"[75] was to form the basis of his science, a science which he claimed was not dependent on the theory of magnitude. As Scholz pointed out, his way of describing crystalline objects with "complexions", possessing "exponents of repetition" ("Wiederholungsexponenten"), showed him the way to a three-dimensional vector calculus with integer coefficients.[76] Without getting too involved in the mathematical details of his treatise, we should name some of J. Graßmann's concepts, which nevertheless had to wait for Hermann Graßmann before they became more precise.

Firstly, J. Graßmann stated that the "product of a combination of two lines"[77] was their point of intersection, that is to say, a view which would later reappear in Hermann Graßmann's planimetric product.[78] Secondly, we will have to mention the peculiar application of the concepts of addition and multiplication to the theory of combinations, even though it never provoked any reaction from the mathematical scientific community.

While, according to J. Graßmann, there was one specific way of adding "complexions", he stated explicitly that there were different ways of multiplying "complexions", depending on what one considered a "unit" to be.[79] This thought had clear repercussions on the work of Hermann Graßmann. It gained precision in exterior vector-algebra, especially in the many multiplicative conjunctions which were developed in this context.[80]

The geometrical theory of combinations represented the mathematical, philosophical and theological center of J. Graßmann's scientific universe. By understanding and teaching it, J. Graßmann proposed a provocative set of ideas which led to a conceptual and methodological reassessment of mathematics: as Radu[81] has shown, this reassessment left ontology, arithmetic and axiomatics behind, presenting a synthetic, constructive, dialectical and comprehensive approach instead.

"I am convinced", he wrote in a footnote to his *Pure Theory of Number* (1827), "that the theory of combinations will be to the history of nature and chemistry what the theory of magnitude is to physics."[82]

The theory of combinations was supposed to be the key to a new synthesis of spirit and nature, of man and God: "One day, [it will be], if I may say so, the connecting link in the galvanic chain between the extremes of mathematics and the science of nature. It will make the chain complete, stimulate the other two links and, thereby, bring completely new phenomena to light."[83]

Justus Graßmann on the formation of crystals:

The Romantic world between analogy in the natural sciences and speculation in the philosophy of nature

When a crystal forms "…this is more than just a manifestation of opposing forces, but a constant oscillation of contraction and expansion, permeating the crystal in certain directions and pulsating in these directions. This equilibrium of fundamental forces … is not motionless and dead, but vibrant and alive. One could even say that it never really exists. Rather, it constantly strives to reproduce itself, and thereby, just like an oscillating pendulum, constantly crosses a certain limit until the counter-reaction becomes strong enough to propel it in the opposite direction, searching for equilibrium. Every element is in continuous undulation. It exists in this movement and, therefore, the whole crystal is in a constant state of *becoming*. … It is not dead, but a beating heart through and through! – We can remark a similar pulsating motion, a similar exchange of contraction and expansion not only in all living beings, but also in the way light, sound, etc. travel. – Its presence is not limited to the formation of the crystal, but it lives on permanently along with the crystal…"[84].

Justus Graßmann was planning to keep up the work in the following years. In case his findings provoked a reaction, he wanted to publish four such booklets per year and, if possible, assemble a group of collaborators around him.

"The second volume", he informed his readers, "should it ever see the light, would not remain confined to such a specialized topic. Rather, in this book, the author is determined to explain his views on the general theory of combinations and its applications in the natural sciences, especially concerning their systematic order. This task would parallel the development of the concept of mathematics, which gave rise to my views."[85]

Unfortunately, since the publication of the first booklet went unnoticed, Graßmann never continued his project. The British mineralogist W. H. Miller was the first to recognize the achievements of J. Graßmann in crystal physics. Without the slightest connection to W. Whewell (1825), J. Graßmann (1829) had used his geometrical method to designate the crystal faces with indices. This method, which became known in the scientific world as the "law of rationality", was rediscovered by Miller (1839).

As an inventor of new theories in mathematics and the natural sciences, J. Graßmann remained an unknown figure.

But, by inspiring and accompanying his son's brilliant scientific achievements, J. Graßmann made a lasting contribution to the development of mathematics. Therefore, his name should not be forgotten.

Closing this section, the table below offers a brief comparison of Justus and Hermann Graßmann's positions and shows some of the similarities, differences and continuities.

Scientific continuities and discontinuities

Justus Günther Graßmann	**Hermann Günther Graßmann**
I. Approaches and concepts in the *Geometry for Elementary Schools* (J. Graßmann 1817, 1824)	
Geometry is a construction in "pure intuition"	Geometry is an application of pure cognitive constructions on exterior spatial relationships, 1844 (A1)
Geometry for Elementary Schools possesses two elements: linear and angular magnitudes. The methodological work begins with lines, continues with angles and, finally, the combination of the two.	Plan for the structure of the first and second part of *Extension Theory*: Part 1: treats affine n-dimensional geometry without metrics, based on oriented *displacements* and exterior products. Part 2: treats the relationship arising from a generalized *concept of angle* and rotation, basing them on the inner product of vectors in the framework of n-dimensional Euclidean geometry. 1844 (A1).

Justus Günther Graßmann	**Hermann Günther Graßmann**
Demands a unitary, stepwise structure for the representation of mathematics.	Presents a stepwise structure for his *Extension Theory*, gradually extending the concept of product and implicitly relying on group theory and the principle of permanence. 1844 (A1).
Rejects the Euclidean way of teaching and representing geometry.	Shares this view, 1844 (A1).
Combines geometry and combinatorics.	Combines *n*-dimensional geometry and combinatorics. Considers vectors and multivectors "continuous complexions". 1844 (A1).
Uses the principle of movement to generate geometrical objects, begins with the point as the "limit of all extension".	Uses the principle of movement to generate geometrical structures of arbitrary (finite) dimensionality through "continuous change of the element" with one fixed direction. 1844 (A1).
Transfers concepts from algebra to geometry; e. g. surface of a rectangle as the product of two adjacent sides.	This was the direct point of departure for Hermann Graßmann in developing vector addition and the exterior product of vectors. 1840 (EBBE).
Peculiar German mathematical concepts.	Same tendency. 1844 (A1).

II. Approaches and concepts in the treatise
***On the Concept and Extent of the Pure Theory of Number* (ZL) of 1827**

Philosophical reflection on the foundations of mathematics.	Philosophical introduction to the *Extension Theory* of 1844. 1844 (A1)
Mathematics must be adequate to its object and serve as a model when applied to the natural sciences.	Expresses the same demand as early as the graduation thesis on the theory of low and high tides. 1840 (EBBE).
Believes that mathematics leads to an understanding of the divine and is a religious practice.	Hermann Graßmann does not express this view explicitly in his work on mathematics and the natural sciences.
Generates mathematical objects constructively by mathematical synthesis in "pure intuition".	Generates mathematical objects constructively by mathematical synthesis in "pure thought". 1844 (A1).
The point of departure of mathematical synthesis is something that is simply "given", abstracted from all real content.	The point of departure of mathematical synthesis (or construction) is something "posited" by thinking, the "particular as such", abstracted from all real content. 1844 (A1).
Mathematical synthesis may be continuous or discrete, the given may be equal or unequal.	Shares this view. 1844 (A1).

<table>
<tr><th>Justus Günther Graßmann</th><th>Hermann Günther Graßmann</th></tr>
<tr><td>Division of the mathematical disciplines:
<table>
<tr><th></th><th>discrete</th><th>continuous</th></tr>
<tr><td>equal</td><td>theory of number</td><td>geometry</td></tr>
<tr><td>unequal</td><td>combinatorics</td><td>?</td></tr>
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<tr><th></th><th>discrete</th><th>continuous</th></tr>
<tr><td>equal</td><td>theory of number</td><td>function theory</td></tr>
<tr><td>unequal</td><td>combinatorics</td><td>extension theory</td></tr>
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<tr><td>Geometry as a mathematical discipline and a representative of the theory of continuous magnitudes.</td><td>Geometry as the theory of physical space. Its mathematical correlate is extension theory. The theory of continuous magnitudes is independent and does not rely on geometry. 1844 (A1).</td></tr>
<tr><td>Natural numbers as the product of a type of mathematical synthesis with no corresponding truth-value.</td><td>Direct constructive generation of whole numbers in the treatise on arithmetic. 1860 (LA).</td></tr>
<tr><td>The concept of number is directly connected to the concept of unit.</td><td>The concept of vector is an aggregate of n independent units. 1861 (A2).</td></tr>
<tr><td>Conceptual differentiation of two elements of a conjunction. Base and exponent in a power are non-commutative.</td><td>Factors in the exterior product of vectors are non-commutative. 1840 (EBBE).</td></tr>
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III. Approaches and concepts in the treatise *On Physical Crystallonomy and the Geometrical Theory of Combinations* (KRY) of 1829

Founder of a *new*, *independent* science, the "geometrical theory of combinations".	Founder of a *new*, *independent* science, "extension theory". 1844 (A1).
Feels the necessity to explain the genesis and development of the theory, and its content, extent and form.	Same approach in the *Extension Theory* of 1844. 1844 (A1).
Begins the "geometrical theory of combinations" without relying on metrics.	Founds extension theory without relying on metrics. 1844 (A1).
"Length and orientation" of lines as the basis for the "geometrical theory of combinations".	"Length and orientation" of displacements as the basis of extension theory. 1844 (A1).
"The point of intersection is the *product* of a combination of two lines."	The point of intersection is the regressive *product* of two straight lines in a plane. This is one of the foundations of the "purely geometrical" theory of curves. 1844 (A1).
The theory of combinations implies the possibility and presence of a large number of multiplicative conjunctions in the "geometrical theory of combinations".	Develops large number of exterior, inner and formal (tensor) products of vectors and multivectors in extension theory. 1844 (A1), 1847 (PREIS), 1861 (A2).

Justus Günther Graßmann	**Hermann Günther Graßmann**
Takes the theory of combinations of Leibniz as a direct point of departure.	Modifies and develops Leibniz' "geometrical characteristic". 1847 (PREIS).
Demands a critique of the foundations of mathematics and a new, better philosophy.	Works on a critique of the foundations of mathematics together with his brother Robert Graßmann, especially in 1847/48 and 1856/57.

2.2 Robert Graßmann (1815 – 1901): Brother and collaborator

From the early 1840s to the early 1860s, Hermann Graßmann was very busy with his research in mathematics and the natural sciences, apart from his political activities. This was a period of intense collaboration with his brother Robert Graßmann (1815 – 1901). Robert was among the people Hermann Graßmann felt closest to, along with Friedrich Schleiermacher and Justus Graßmann. All three played an important part in shaping Hermann Graßmann's worldview, as well as his political and scientific development. It is impossible to understand his life and achievements without also understanding the relationship between the two brothers.

As in the cases of the von Humboldt or the Grimm brothers, Hermann and Robert Graßmann – leading their lives in Stettin and quite isolated from scientific life – were partners, inspiring and criticizing one another while each brother developed his worldview, his political and scientific positions. Concerning their characters and temperaments, they were completely different. Hermann was persistent, diligent, timid, modest, completely immersed in scientific work and not a good socializer. Robert was very aware of outer appearances, a practical personality, very involved in political and economic life, sure of himself and not exactly a modest person. Nevertheless, for the most part of their lives, both brothers were united in friendship and scientific collaboration.

As early as 1835, during his days as a student in Berlin, Hermann Graßmann wrote the following lines to his twenty year-old brother, referring to a disagreement between the two: "And therefore we … will learn from each other and adopt what we have to offer to one another intellectually, but without agreeing to something outwardly when we are not completely convinced inwardly."[86]

This mindset, which expressed the potential and the limits of a collaborative research effort, was to become the foundation for subsequent studies in the natural sciences, mathematics, philosophy and politics. This is how Robert Graßmann remembered the collaboration in 1896: "The two of us were intimate friends, and for a number of years we spent hours working together, every day, in the fields of philology, philosophy, mathematics and physics."[87]

Robert Graßmann, born in 1815, spent the first years of his life with his uncle, school councilor Friedrich Heinrich Graßmann, whose marriage had been childless. Later, Robert claimed to have already spent two to three hours every day studying mathematics and physics when he was an advanced secondary-school student. After having graduated from "Gymnasium", he took up university studies in Bonn in 1834. The results from his first semester were so promising that he was made a member of "Seminar for the entire natural sciences".[88]

We may assume that the death of his sister Alwine in November of 1834 affected him strongly, for he radically turned his back on the natural sciences and concentrated almost completely on theology. Only in the third and fourth semester did Robert Graßmann slowly focus on other subjects again, attending classes in the history of philosophy, logic, psychology and anthropology. In contrast to his brother, who traveled little, and thanks to financial support from his uncle, Robert went on long trips through Germany, Switzerland, Italy, Austria, Belgium, the Netherlands and France. After having completed his studies in Bonn, he spent another two semesters at Berlin University, studying almost exclusively theology and mostly attending lectures by August Wilhelm Neander.

According to documents left behind by Robert Graßmann, he studied mathematics with Carl Dietrich von Münchow, the natural sciences with Ludolph Christian Treviranus and Heinrich Wilhelm Dove. He also learned about philosophy from Christian August Brandis and Immanuel Hermann Fichte, while also attending theology classes by Neander, Friedrich Bleek and Karl Immanuel Nitzsch.

In 1838, he passed his first examination in theology and returned to Stettin.

Back home, now working as a teacher, he concentrated on mathematics and the natural sciences and got in touch with J. A. Grunert, who was the president of the recently established Pomeranian scientific examination commission in Greifswald. Grunert advised Graßmann on further reading for his autodidactic studies of mathematics: Lacroix, Ettinghausen and Cauchy for analysis, Biot for analytical geometry, Francoeur for mechanics and Lagrange for the theory of analytic functions.

While he resumed his studies, he taught for three months at the Stettin seminar for teachers, standing in for another colleague and then, during another three months in 1839, for the director of the seminar. He also served in a military division of engineers (1838/39), where he invented a galvanic fuse for mines.[89]

After his release from the army – his brother had been exempted from military draft due to his weak physical condition – as an engineering corps officer, he tackled the examinations he had to take in order to become a teacher. His brother Hermann introduced him to the work of Lagrange. Robert Graßmann was in a hurry and had many other obligations.

On 28 December 1839, Grunert presented Robert's application for an examination in mathematics, physics and philosophy to the commission.[90] Ernst Stiedenroth, professor of philosophy at the University of Greifswald, was responsible for the examination in philosophy. He asked Robert Graßmann to specify and criticize the reasons supporting the view that mathematics could not be a philosophical science.

Robert Graßmann immersed himself completely in this problem. He had to ask the commission for additional time twice, and his final paper, over a hundred pages long, remained unfinished. Robert submitted his examination theses on 24 July 1840. Unfortunately, they have been lost. Stiedenroth was enthusiastic about Robert's originality as a philosopher and his general competence. Even though Grunert still remarked certain gaps in Robert's philosophical knowledge, he gave him full permission to work as a teacher. Robert Graßmann had been the fifth candidate of the newly founded commission, and he had passed brilliantly, receiving the unconditional permission to teach mathematics, physics, philosophy and theology. Following his brother, he became a teacher in 1841.

He began his teaching career at the "Friedrich-Wilhelmschule", before receiving a permanent position at the "Höhere Töchterschule" in Stettin, where he would remain until his retirement in 1852.

Even though both brothers had felt very close to each other since their childhood days – we have already seen how hard it was for Hermann Graßmann to leave his friends and family to work at the trade school –, scientific collaboration began only around 1840. At this point in time both brothers had become teachers and turned to scientific work in their free time.

All in all, it was a surprisingly asymmetric situation.

In his first examination on the way to becoming a teacher in 1831, Hermann Graßmann had not received full permission to teach mathematics in all grades. Concerning philosophy, the diploma said that he had "incomplete knowledge about the results of philosophical investigations" and that he therefore was "still incapable" of "teaching philosophical propaedeutics". Nevertheless, the document stated, he possessed "the kind of philosophical education necessary for developing general concepts methodically"[91]. He had to submit himself to a second round of examinations (1839/40) in order to attain the unconditional permission for teaching physics and mathematics. This time, he passed the tests so brilliantly that his graduation paper, *Theory of Low and High Tides*, became the cornerstone of a new mathematical discipline.

Robert, in contrast, did not have to try twice. While his knowledge of mathematics was incomplete, his philosophical graduation thesis on the relationship of mathematics and philosophy was highly praised.

In 1840, the scientific paths which both brothers were destined to take were slowly materializing: on the one hand, Hermann, the philosophizing mathematician, on the other hand, Robert, a philosopher inspired by mathematics.

In 1839, Schleiermacher's *Dialectic* (DIAL) was published posthumously. Even though both brothers were very busy teaching, they immediately began to read the book in 1840, studying it together. After all, Hermann had been fascinated by Schleiermacher ever since his days as a student.[92] By reading Schleiermacher with his brother, Hermann added philosophical and methodological knowledge to his mathematical abilities. He acquired the tools he needed and, at Easter of 1842, he began working on *Extension Theory*.[93]

Robert Graßmann felt equally inspired by Schleiermacher's thoughts. *Dialectic* prompted him to begin working on a philosophical *Theory of Thinking* during the winter of 1844/45. This theory was supposed to provide a foundation for all scientific activities.

In the following years, the two brothers continued their philosophical studies. In 1846 they worked on Hegel's dialectics, but remained skeptical of its speculative nature.[94]

Robert Graßmann on Hegelian philosophy (*Theory of Thinking* 1875)

"The Hegelian school of thought is the summit and, as it seems, also the end of these chaotic groupings and systems. Uniquely presumptuous, it rejects the procedures and results of the exact sciences. Only modern philosophers could be so blind to believe that the empty babble of their school captures the essence of things. With arrogance bordering on madness, they scoff at the Revelation and divine teachings, only to create God through formulas from the nothingness of their thinking. This school of thought has perpetrated unspeakable evil and is the reason why so many people are skeptical of philosophy. This has had dubious consequences for recent intellectual developments, and it has strengthened the rule of half-truths and empty talk."[95]

After having read Hegel, Robert was convinced once and for all that a new philosophy, even a reconstruction of the entire edifice of knowledge, was necessary. "[W]orking almost daily for 4 to 8 hours"[96], he invested over half a century of efforts into this encyclopedic project.

Hermann Graßmann's discoveries formed the basis for their brotherly mathematical discussions. Hermann composed the first (and only) part of his *Extension Theory* (A1) in a short period of time, ranging from Easter 1842 until autumn 1843. Even though he was under a considerable strain due to his daily work as a teacher, he created different versions of his masterpiece, "now in the Euclidean form for the greatest rigor in demonstrations and theorems, now in the form of a narrative development for the greatest clarity of arrangement, again with both woven together"[97]. In 1843, he presented his findings to a circle of close friends.[98] Of course, Robert Graßmann also participated in the discussions.

In 1844, Hermann Graßmann published *Linear Extension Theory*. Contemporary mathematicians ignored him almost completely. The philosophical nature of Graßmann's approach probably also played a part in this lack of appreciation. Obviously, this raises a number of questions.

Did Robert urge his brother to choose *this* particular philosophical form for *Extension Theory*?

Had Robert been the philosophical mastermind during the genesis of *Extension Theory*?

Had the main inspiration *not* come from Schleiermacher, as some remarks by Hermann Graßmann might make us believe, but from Robert and his philosophical examination thesis?

We might go even further and ask the following question: Even though Robert's paper on the philosophy of mathematics was highly praised, the most important contemporary philosopher of mathematics was Jakob Friedrich Fries. Did *Fries* influence the style of the first *Extension Theory*?[99]

We will explore these questions briefly.

In his "Picture of my Life", which appeared in the introduction to the *Edifice of Knowledge* ("Gebäude des Wissens"), Robert Graßmann explained that he had dedicated the greatest part of his studies to philosophy. "Among the ancients, he studied Aristotélēs and Plátōn, and among the moderns, mainly Hegel and Schleiermacher."[100] If Robert Graßmann had read Fries, he probably was not overly impressed. R. Graßmann supplemented his *Edifice of Knowledge* with a "History of Philosophy", the structure and contents of which confirm this assumption.

"The following history of philosophy, which I offer as an introduction to the theory of knowledge, aims to outline the teachings of individual philosophers in reference to the theory of knowledge", he wrote there, " …whereas the latter will mostly ignore the fictions and illusions of the philosophers, which only served to fill the gaps created by incomplete knowledge and which were mostly worthless for the theory of knowledge and philosophy."[101]

R. Graßmann continued by giving a selective recapitulation of the historical development of philosophy, and of knowledge in the natural sciences and mathematics. This section probably contains elements from the lost examination treatise. Again, Fries is not mentioned in this history of philosophy. There is, however, a section on Schleiermacher. R. Graßmann called him "the most important critic … we have seen in recent times."[102]

"Schleiermacher's great merit", he explained, "is that he was the first to truly grasp and introduce into science a theory of scientific discovery or speculation as the highest branch of the science of logic. He has this merit even though his idea remained stuck in a theoretical stage and even though Schleiermacher did not yet know how to use it to reorganize the sciences."[103]

And Robert went on to say: "According to his 'Dialectic' Berlin 1839, only two academic fields can show us the idea of knowledge. Both of these fields deal with the idea of knowledge, that is to say, the mutual relationships between thinking and the existent. Dialectics, which deals with the oppositions within unity, does so in the conceptual frame of the general, whereas mathematics, which only deals with equal and unequal magnitudes, does so in the conceptual frame of the particular. According to him, every real thought contains as much science as it contains dialectics and mathematics (§§ 344 – 346). Mathematics is closer to the empirical, dialectics closer to the speculative form. The empirical process always precedes the speculative process, contextualizing it. Schl. is completely on the mark in these theorems; but, as he remarked himself, he lacked knowledge of mathematics ..."[104]

These extensive quotations prove that Schleiermacher was a *major* influence on Robert Graßmann's concept of philosophy. In turn, his brother Hermann had been responsible for this. He had been the one who had inspired Robert to study Schleiermacher's *Dialectic*.

In their understanding of mathematics and philosophy, both brothers relied directly on Schleiermacher, though they both took up different aspects. It seems as if one sentence could sum up the decisive difference between the brothers' views: "The empirical process always precedes the speculative process, contextualizing it."

Hermann would probably not have approved. In Hermann's view, going back to his father, both processes interlocked. By synchronizing the methods of representation and research, both processes ensured the organic wholeness of the generated knowledge and the subject's sovereign overview of his object.

Robert relied on Leibniz and followed his own interpretation of Schleiermacher: he founded his philosophy on what was given and an empirical precondition, language; then he continued with mathematics (as a theory of forms) and logic, constructing an atomistic monadology on this foundation and finishing his system with the realm of God and His angels.

Philosophically and methodologically, the *Extension Theory* of 1844 essentially relied on Schleiermacher's and Justus Graßmann's philosophical and dialectical intentions. Only after the mathematical public disapproved of the work did Hermann seem to have accepted his brother's concept.

In his memoirs, Robert wrote the following on the collaboration of the two brothers:

Hermann "was working on his treatise on low and high tides during the winter of 1839 and 1840. ... [He found] a number of laws concerning the addition and multiplication of lines or displacements. During the following years, he kept working on this and published 'Die lineale Ausdehnungslehre' in Leipzig in 1844. In this work, he mostly relied on geometrical magnitudes, or static elements, and tried to use these to elaborate the laws of this new science. ...

In 1847, the two brothers Hermann and Robert Graßmann joined forces to derive extension theory, independently from geometry and following the strict procedure of the development of forms, as an independent branch of pure mathematics and to extend it until it reached the limits of its validity. … Each one of the two brothers felt that, alone, he did not have the strength to think every idea through for every single operation, and that the whole affair could only be mastered by two minds. … Robert Graßmann's merit is to have contributed to the general validity of the approach and the rigor of form, thereby also contributing to the solution of the problems in question."[105]

Obviously, Robert did not want to take responsibility for the form of presentation of the *Extension Theory* of 1844. In fact, he remained skeptical of its form, while subsequent mathematical works, such as the textbook of arithmetic (LA) published by Hermann, clearly bore the mark of his conception of formal rigor.

Carl Gottfried Scheibert, Hermann's brother-in-law who had published a *Textbook of Arithmetic and Plane Geometry* for secondary schools in 1834 and who was "totally in accord"[106] with Justus Graßmann's textbook of trigonometry (1835), wrote the following words when Hermann Graßmann's textbook of arithmetic was published in May of 1861:

"But I am not at all inclined to learn more about the way mathematics has been developing recently, for it is a path leading away from school. The textbook written by my brother-in-law, Professor Graßmann in Stettin, is a typical example. I will try to put it mildly: This way of thinking seems to turn a mind into dried fruit, … while I consider systematical learning, which progresses along the lines of dialectically developing concepts, to be the pedagogical approach to mathematics. I take the ability of solving mathematical problems to be utterly insignificant in intellectual education and utterly inappropriate when it is supposed to show that somebody has received this education."[107]

Apart from the conceptual transformation of what was considered to be the essence of mathematics and of the mathematical method, just the work of restructuring mathematics and founding extension theory was an immense achievement for the two brothers.

In 1890, Robert characterized their collaboration in the following way: "In 1847, the two brothers Hermann and Robert Graßmann joined forces to derive extension theory, independently from geometry and following a strict development of forms, as an independent branch of pure mathematics and extended it until it reached the limits of its validity. The two composed a booklet of 132 pages, which the author still has among his possessions. They began this collaboration by treating the theory of domains until reaching domains of n-th order and proving their independence from the original units. At this early stage, they found the theorem which stated that the sum of the orders ("Stufenzahl") of two domains is equal to the number of orders of the sum-domain ("umfassendes Gebiet"), plus the number of stages of the intersection domain ("beiden gemeinschaftliches Gebiet").

Then they derived the different kinds of multiplication and found the three types of multiplication: flattening ("Flachung"), which they called exterior multiplication, shadowing ("Schattung") or inner multiplication, and additive (algebraic) multiplication.

In flattening, or exterior multiplication, they began by establishing the laws of flattening, namely the laws concerning the change of sign when the flats ("Flache") or factors are interchanged, as well as the laws of linear change. Then they treated the product of two higher-order magnitudes or the flats of the flats and the sum of these flats and developed the laws of relational flattening ("bezügliche Flachung"), as well as the laws of the complement to the principal domain and the laws of reduction to a domain.

In inner multiplication (shadowing), they established the relationship between inner multiplication and the multiplication of the complement, as well as the theorems stating under which conditions the inner product will equal zero and the theorem on the equality of inner squares, or, in other words, the theorem which introduces angles.

Finally, concerning additive multiplication or weaving ("Flechtung"), they developed the corresponding laws, introduced quotients, derived the laws for division, for the affinity and multiplication of quotients. They also discovered the powers of quotients, as well as products with one or multiple gaps and a polynomial theorem for this kind of product."[108]

Robert Graßmann's explanations, which even outdo Hermann Graßmann's peculiar German terminology, show very clearly how much progress H. Graßmann had made in 1847 in his new mathematical discipline and how closely the two brothers were connected – at least until 1848 – in their common scientific work.

The scientific collaboration of the two brothers was not a unilateral affair. Hermann Graßmann inspired his brother by pointing him towards Schleiermacher (and towards the unity of the natural sciences and philosophy). He raised Robert's awareness for the necessity of an analysis of the foundations of mathematics, especially of geometry, by developing his *Extension Theory*. Robert, in turn, paid his brother back by uncovering logical fallacies in mathematical definitions and by checking his brother's mathematical reasoning. Robert always strove for the highest possible level of generality in mathematical thinking.

The political upheavals of 1847 suddenly intruded on the two brothers' research projects. Just like other parts of the country, Stettin was affected by violent protests against food shortages, while the bourgeoisie demanded more constitutional rights. They reacted by reading Schleiermacher's *Theory of the State* ("Lehre vom Staat", 1845) and Dahlmann's *Politics* ("Politik", 1835). By founding their own newspaper, in support of the monarchical and conservative side, the two brothers actively took part in the political struggles of 1848.

Eight years later, the Graßmann brothers returned to scientific collaboration, which by then had become more sophisticated. Robert Graßmann later explained that they "gave up on their method of working together". He went on to say: "They limited themselves to brotherly conferences. One of the two would come up with a draft and give a presentation; the other would quiz him critically and make objections. This method was extremely fruitful and inspiring. It quickly led them to the strictly scientific method in the individual branches of the theory of forms or mathematics, as well as logic. This work method quickly brought to light Hermann's preference for the mathematical branches, for the theory of numbers and extension theory, while Robert, the other brother, was more interested in the philosophical branches, in logic and the theory of connections or combinations. Therefore, by late 1856, the two brothers practiced a division of labor. Hermann dealt with the theory of numbers and extension theory, Robert with logic and the theory of combinations."[109]

The collaboration showed results with the *Textbook of Arithmetic* (LA), published in 1860 by Hermann Graßmann, and the second version of *Extension Theory* (1862, A2), which followed a strictly Euclidean form.

Robert, who in the following years concentrated on politics, waited until 1872 before publishing his summary of their collaboration in the book *The Theory of Forms or Mathematics* ("Die Formenlehre oder Mathematik", R. Graßmann 1872). These three books are manifestations of Robert Graßmann's ambition of attaining the highest possible level of generality and, in his results, of formal and symbolic abstraction. Unfortunately, explicit explanations concerning the philosophical and dialectical aspects of the objects in question had fallen by the wayside. By 1872, both brothers had made major progress. Robert Graßmann's book *The Theory of Forms or Mathematics* (1872) made him a pioneer of mathematical logic[110] and a precursor of the reconstruction of elementary mathematics.[111] In contrast, his enormous, ten-volume encyclopedic and philosophical *Edifice of Knowledge* ("Gebäude des Wissens", R. Graßmann 1882 – 90) remained insignificant from a scientific point of view.

Even if we left aside *Extension Theory*, Hermann Graßmann's achievements in the *Textbook of Arithmetic* (LA) are simply groundbreaking. Leaving M. Ohm[112] far behind, H. Graßmann presented a concise and new foundation for arithmetic.[113]

Even though Robert Graßmann had become an indispensable partner in Hermann Graßmann's scientific development, he still could not make up for Hermann's lack of contact and interaction with the most important contemporary mathematicians. In fact, Robert Graßmann was one of the factors hindering the public recognition of Hermann Graßmann's mathematical ideas because he supported Hermann's peculiar German terminology, even surpassing him in this scientific nationalism and thereby increasing their isolation from mathematics at the time.

In this context, some remarks on Robert Graßmann's worldview are appropriate.

As we have already seen, the interests of the two brothers began to diverge slightly in the 1850s. The political events of 1848/49 certainly played a part in this. While Hermann Graßmann limited his focus on mathematics and the natural sciences, Robert Graßmann increasingly turned towards politics, philosophy and logic. Nevertheless, they kept up their discussions for many years.

Even though it is impossible to measure Robert's philosophical influence on his brother point by point, Robert's convictions often reappeared in a more moderate form in the writings of Hermann Graßmann.

Concerning Robert Graßmann's worldview and political positions, an essay on Ernst Moritz Arndt, written by Friedrich Engels in 1841 (Engels 1967b), is especially interesting. What makes this essay so useful is the fact that Arndt, in his opinions and his territorial, social and economic surroundings, had much in common with the Graßmann brothers.

Friedrich Engels explicitly highlighted the connection in Arndt's views between theological orthodoxy, German nationalism and political theory in late Romanticism. Engels interpreted this connection as a petit-bourgeois reaction to the War of Liberation of 1813 – 1815. Engels emphasized that German nationalism was mainly directed against French cultural hegemony, but he also pointed out that its fascination for the "primeval German purity"[114] also favored a reactionary mindset. Concerning Robert Graßmann, this certainly was the case. In his 1875 *Theory of Thinking*, he wrote: "When the French language came to Germany, so did French carelessness and immorality, ruining our aristocracy and princes."[115] For R. Graßmann, extreme nationalism was the only possible consequence:

"...for the schools, we demand German teachers, who have been touched by the truly German spirit and moved by the wind of German enthusiasm. We demand men who speak German, who think German and who are capable of introducing the children to the German language and the German temperament."[116]

When he went on to say that "[i]n German, we can find a German word for every concept imaginable, and it is the duty of scholars and teachers to find these German words and establish them in the language"[117], this was the root of his obsession of expunging foreign terms from scientific writing. It was this tendency which made it extremely hard to study the works of Hermann and Robert Graßmann and which, in the end, was responsible for the lack of public appreciation they both had to accept.

How could these tendencies become so influential in R. Graßmann? The answer lies in the economic underdevelopment of his surroundings and the scientific and ideological narrow-mindedness of the petite bourgeoisie, which, in this context, was determined to demonstrate its social weight by celebrating its "inertia"[118].

Apart from R. Graßmann's political views, his studies in mathematics and the natural sciences also resonated in his philosophical constructions. Firstly, R. Graßmann

Robert Graßmann and the Germans

"The German Reich is a centrally located country and nevertheless, cut up by mountain ranges, it is at the same time the land of diversity and free progress. The Germans are tall and strong, diligent and well-organized, loyal and trustworthy, interested in their inner lives and therefore good-natured and mild. The German nation represents deep spirituality, classical music, and Germans are also men of science. In both respects they are the teachers of foreign nations, the true keepers of scientific and religious progress, a nation of thinkers and scientists, unrivalled by any other nation in the depth of their research, the universality of their knowledge, the sternness with which they struggle to solve the holiest of questions. No other nation equals their intimate spirituality and the melodies and harmonies of their music..."[119]

transferred a mindset influenced by mathematics and the natural sciences to philosophy. This showed itself in elements of empirical realism, extracted from the natural sciences: "The path we will have to follow ... is no other than the path which the great natural scientists, Copernicus and Kepler, Galilei and Newton, took. Once we have chosen this path, it will lead us to our goal, and necessarily so, for it is the path on which theorems from the natural sciences, obtained by empirical observation, become part of the reflective process of the mind. Just like every natural scientist, we begin with experience, obtaining theorems reflecting this experience through observation and testing, we correct errors, and then we compare our theorems and laws and look for patterns. In every one of these steps, we are completely in accord with the procedures of the modern natural sciences, and this is the only possible way to attain reliable and generally accepted knowledge. Therefore, we demand of every philosopher that he must follow this path if he wants to gain knowledge. And every theorem that has not arisen from experience is an unscientific theorem, a product of fantasy and devoid of all scientific value."[120]

Secondly, R. Graßmann gave an orthodox and theological interpretation of Schleiermacherian – and, to a lesser extent, Hegelian[121] – dialectics. This interpretation sought to connect dialectics to the natural sciences through a certain kind of mathematical idealism: "The human mind must retrace the *spiritual forms and laws which it finds in nature*, recreating them in order to understand them. Every natural scientist must therefore know the theorems of the theory of space and of numbers, in short, of mathematics, just to be able to perceive and understand the laws of nature in the first place ... [emphasis mine – H.-J. P.]"[122].

Taking Schleiermacher[123] as a point of departure, R. Graßmann defined God as the "deepest origin of all being and becoming ... the deepest origin of all knowledge and understanding ..."[124]: "All change, all life in the universe arises from the one and only God. All oppositions and chiasmata, which transform unity into diversity, arise from Him. ...God creates from his free, immaterial will ... creatures whose essence in the external world is divine, which obey His laws in their movement through empty space and therefore are external manifestations of His laws ..."[125] But Robert Graßmann's concept of God was not merely pantheistic or deistic. Rather, in an orthodox and theistic sense, he declared God to be the active, guiding hand of all worldly events:

"So, at a given moment and at a given place, God intervenes", R. Graßmann pointed out. "He is not the inert, passive idol of the philosophers' fictions, but He is active, creative and in motion. He is a living and spiritual God, who gives the things their essence and their movement, who gives us humans spirit and life, and gives all other, simpler things the force of existence and causality, an eternal, immortal and inalienable heritage."[126] Even though R. Graßmann was a theoretical thinker with a great gift for logical coherence, he also strayed into irritating and simply weird fantasies about the current activities of God on earth.

Leibniz had been one of the key figures for Justus Graßmann's theory of combinations, Hermann Graßmann's *Extension Theory* and Robert Graßmann's logic. From a mathematical point of view, the insights of Leibniz had been useful and also sparked a number of ideas in different academic fields. Now, Robert Graßmann created his ontology from a monadological point of view. According to R. Graßmann, God, a simple point-creature or monad, had created all other creatures – mass-creatures, ether-creatures and spirit-creatures. Mass-creatures, which were more or less identical to contemporary concepts of particles in physics, belonged to the realm of necessity. Spirit-creatures – human souls or "angel souls" – participated in the realm of freedom and chance. This is how R. Graßmann tried to unite mechanical determinism and personal freedom. At the same time, he modified Newtonian general laws and applied them to the spirit-creatures, the immortal souls. Physicalism and speculation became intertwined in a specific way, and R. Graßmann had to give up on his initial principles of scientific work.

Robert Graßmann and the theory of the Divine world

In the introduction to his *Edifice of Knowledge*, R. Graßmann gave the following explanations of his version of a science of God: "The science of God is a strictly scientific introduction to the Divine teachings and the Divine world. ... *The theory of the Divine world* proves the existence of God, explains His general characteristics and then leads us into the realm of ether-creatures and ether-spirits in heaven. ...the *theory of the Divine world* will be the theory of the heavenly life of mankind and will offer strict proof of our immortality."[127]
"Every second, on all the different earths, God creates 30.000 million spirited creatures or humans, every year a trillion humans, and He has been doing so for thousands of years", he wrote in a different section, meditating on the creation of millions of earth-like planets and even more human beings. "But even before this epoch, God was not idle, but created innumerable stars and creatures."[128]

But this monadology also gave rise to an impressive case of hypothetical construction in the philosophy of nature: in 1862, Robert Graßmann published his and his brother's ideas on atoms and the ether.[129] Unfortunately, Hermann Graßmann's specific contributions to this model are hard to discern.[130] Two aspects of the brothers' concept of the ether and atoms are worth mentioning in this context:

1. The ether is an imponderable particle-gas. The ether particles consist of a positively and a negatively charged electric elementary particle, which revolve around each other.
2. Atoms are not the smallest elements of nature. They consist of a nucleus (mass particle) and orbits of positively and negatively charged electric elementary particles. The orbits determine the chemical and electric properties of the atoms, that is, the molecular properties of bonds and dissociation behavior. The orbit determines the spatial dimensions of the atoms.

Friedrich Kuntze made the following remarks on this point: "Here we can see that, in 1862, about twenty years before Helmholtz' speech on Faraday, the Graßmann brothers taught the atomistic structure of electricity. And we can also see that, according to the Graßmanns, atoms were not the smallest elements in chemistry, but also that they did not recur to the traditional explanation which stated that atoms consisted of hydrogen atoms or particles of such atoms, or – generally speaking – of material elementary particles. Instead, they used elementary quanta of electricity to explain the structure of atoms and their chemical properties."[131]

In many ways, the Graßmanns' hypotheses on atoms were decades ahead of similar models in physics, but they went unnoticed in physics at the time.

Finally, we will have to remark that R. Graßmann's eclecticism is unsettling and fascinating at the same time. On the one hand, he gave extremely strange explanations on the physical properties of "angel souls", on the other he developed fascinating atomic models.

Despite speculations about the high number of worlds created by God and the corresponding number of Saviors, R. Graßmann also expressed interesting dialectical thoughts: "We call two concepts or things, which are fundamentally different and form two elements of a unity, an opposition", he wrote in 1875. "Therefore we can say that it is the task of philosophy to search for the oppositions in unity, and the unity in oppositions."[132] R. Graßmann even brought forward a type of realism which bordered on materialism, as in the following statement: " ...if we call space and time, in which beings interact, *reality*, then we can also say that beings are real, interacting things. In contrast, concepts are intellectual representations of these real things, and they exist only in the mind, not in space, only in thinking, not in reality."[133]

In philosophy and the theory of science, Robert Graßmann clearly opposed every perspective favoring a purely subjective construction of the world in the human mind.

He rejected this position for its arrogance and because, in his view, it meant rivaling God's creation of the world: "It certainly is a sublime task", he wrote, "when the modern theory of science aims to create the entire world of spirit and reflection from the nothingness of thought. In the domain of thinking, this is as if, in the domain of external action, man aimed to create from his bodily nothingness without any material, without any tools and strength, the entire world of physical objects: suns, planets, and earthly creatures."[134]

Robert Graßmann's philosophical views arose from a confusing synthesis of spontaneous materialism, inspired by the natural sciences, the Romantic dialectics of nature, Fideism and classical bourgeois philosophy. A provincial flower on the tree of knowledge, this philosophical position turned to the past in order to find answers for the present. Nevertheless, in a limited context, it was capable of exerting considerable influence on research in mathematics and the natural sciences.

2.3 Friedrich Schleiermacher's impact on Graßmann and fundamental thoughts from his lectures on dialectics

Friedrich Daniel Ernst Schleiermacher was one of the most remarkable personalities of early 19th century Germany. He was a theologian, philosopher, pedagogue and politician. His historical epoch was one of gradual political changes, implemented by the German bourgeoisie. The bourgeois Revolution in France (1789) and the Industrial Revolution in England announced the end of feudalism and the triumph of capitalism. As a consequence, a unique historical period of change shook the intellectual realm of ideas in which the greatest minds of the Enlightenment, of Classicism, Romanticism and Neo-Humanism were fighting for supremacy.

Even though Schleiermacher's philosophical work is as abundant as his theological writings, his achievements were received very differently. He was hailed as a "19th century Father of the Church"[135], but, as Fischer remarked, his philosophical work remained "astonishingly ineffective"[136], even though Schleiermacher's system of the sciences "was as sophisticated as any other concept from German Idealism"[137].

But the Graßmann brothers reacted very differently to Schleiermacher.

"As a man of feeling, Schleiermacher was pious. As a man of thought, he was godless"[138], Robert Graßmann wrote in his "History of Philosophy": "Needless to say, his theory of God therefore turned out to be an incoherent, unscientific structure …"[139]

In contrast, both brothers valued Schleiermacher as a philosopher. As we have already seen, in his curriculum vitae of 1833, Hermann Graßmann claimed that he owed Schleiermacher infinitely in matters of the spirit, because Schleiermacher had taught him "to approach every branch of science competently. He did this not by giving positive elements, but by showing how it was possible to approach every investigation correctly and continue it independently, thereby enabling the scientist to find the positive elements on his own"[140]. Generally speaking, though with less biographical enthusiasm, Robert Graßmann shared this view.[141]

Surprisingly, the words of Herman Fischer, one of the editors of the German edition of Schleiermacher's collected works, capture the Graßmanns' position concerning Schleiermacher's philosophical views very well:

"His dialectical style of thinking", Fischer says, "enabled him to relate every phenomenon to something else, to contextualize phenomena scientifically and interpret them as parts of a larger whole. He rejected all radical alternatives and absolute oppositions. Every theorem potentially contained its opposite. Schleiermacher's dialectics was not an exclusive, but an inclusive method, and he made it into the foundation of an impressive structure in the theory of science. By connecting recognition and action, or, as Schleiermacher also put it, symbolic and organizational action, the categories of the individual and the general, of speculation and empirics, he showed the reader how problems were connected. Schleiermacher used the process of relating speculation to empirics as a guiding principle of philosophical reflection. In the movement of German Idealism, this was exceptional. Schleiermacher's way of solving problems remained influential even after the epoch of speculative philosophy had ended ..."[142]

Schleiermacher was a fascinating personality. Not only Hermann Graßmann, even the young Friedrich Engels was impressed. Schleiermacher was not simply an exceptional personality at a time of groundbreaking changes, "he embodied an intellectual and cultural position, he stood for an entire epoch."[143]

The life and works of Schleiermacher are like a complement, like a contrasting alternative to Graßmann's provincial lifestyle. Nevertheless, Graßmann felt that he could relate to Schleiermacher, and this feeling was one of brilliant inspiration.

Schleiermacher's career is like "the other side" of Hermann Graßmann's life. In his work, Schleiermacher had been moved by many feelings and trends: the Romantic philosophy of nature, the resistance against Napoleonic foreign rule, the movement for school reform, individualism and intimate religious feelings, to name just a few. Schleiermacher's thinking had already been closely intertwined with the work of Justus Graßmann. It changed Hermann Graßmann's views of life and influenced Robert in his. The brothers were inspired by Schleiermacher's self-confident Neo-Humanistic liberalism, which maintained a connection to the natural sciences through dialectic thinking without severing its ties to a deep feeling of religiosity.

Schleiermacher is an integral element of Graßmann's biography. Therefore, we will begin by giving a rough sketch of Friedrich Schleiermacher's life and works.

Then, we will have to determine why Schleiermacher's dialectics was predestined to exert a substantial influence on Hermann Graßmann's concept of mathematics. The point is that we are dealing with a unique historical case: *a direct influence from philosophy on the reconstruction of a scientific discipline*.

The life of Schleiermacher

Friedrich Schleiermacher was born in Breslau on 21 November 1768. He was the oldest son of a Prussian Reformed military chaplain. From age 14 to 19, he and his brothers were educated by the Moravian Church ("Herrnhuter Brüdergemeinde"), a religious movement which had arisen from Pietism and the Bohemian Reformation. In 1787, Schleiermacher left this theological community, revolting against its dogmatic rigidity and strict system of surveillance. He began studying theology at the University of Halle, where Enlightenment thought was very much alive. After having completed his two years of studies, Schleiermacher had to prepare his examinations in theology and search for a teaching position in a private household. Since such a position was nowhere in sight, he stayed with his uncle S. E. T. Stubenrauch. After having passed his examinations in theology in 1790, he became a private teacher in the aristocratic von Dohna household, in the small town of Schlobitten. Initially, Schleiermacher and the family got along well. But, in 1793, his pedagogical methods first provoked a conflict, and finally an open disagreement with Count Dohna. Schleiermacher had to return to his uncle's house in Dresden and began to prepare his second theological examination. In 1784, he became a minister's assistant in Landsberg[144] and resumed his work as a private teacher. His pedagogical work "was partly inspired by the new methods of Rousseau and the teachings of the French revolutionary movements ..."[145] In 1796, he took up a job at the Berlin Charité, becoming the hospital's chaplain. The following years in Berlin influenced Schleiermacher strongly.

In Berlin, Henriette Herz, wife of the renowned medical doctor Marcus Herz, introduced Schleiermacher to the city's cosmopolitan and wealthy Jewish circles. These were private meetings in which brilliant bourgeois minds met: people like Schiller, Fichte, Jean Paul, W. von Humboldt, the Schlegel brothers, Arndt, Mirabeau, Steffens and Mme de Staël. Schleiermacher became a close friend of Friedrich Schlegel, and the two shared an apartment for a while. Friedrich Schlegel's personality fascinated Schleiermacher and, for a longer period of time, he was strongly attracted to German Romanticism. In 1799, he published *On Religion. Speeches to its cultured despisers* ("Reden über die Religion"), reacting to the great "Controversy of Atheism", which in the late eighteenth century concerned the allegedly subversive philosophical views of Johann Gottlieb Fichte. A year later, the *Soliloquies* followed ("Monologe", Schleiermacher 1914a). Since his theological superiors disapproved of his connections to the Romantics, attacked his atheistic tendencies and frowned on his friendship with a colleague's wife, Schleiermacher accepted a position as a preacher in the town of Stolp in 1802. In this state of isolation, Schleiermacher worked on his first, strictly scientific piece of writing, the *Foundations* *of a Criticism of Traditional Moral Doctrine* ("Grundlagen einer Kritik der bisherigen Sittenlehre", 1803). But since, in the long run, being a preacher in rural surroundings was not enough, Schleiermacher applied for a teaching position at the universities of Würzburg and Königsberg. As a consequence, the Prussian government made him a professor and preacher at the University of Halle. He quickly became an intimate friend of the natural scientist and philosopher of nature Henrik Steffens. When the Napoleonic army shook the foundations of the Prussian state in 1806, Schleiermacher became politically active. His sermons took on a political tone and, when Halle came under Napoleonic rule, Schleiermacher returned to Berlin. Based in Berlin, he got in touch with Stein, Scharnhorst and Gneisenau, and in 1808 he traveled to Rügen and Königsberg with secret missions. He met Bülow and fought to rejuvenate Germany by bourgeois means, hoping to shake the Napoleonic rule and spark a national uprising. He began giving secret lectures in Berlin, along with Fichte, Wolf and Schmalz. For the first time, political theory became part of his repertoire.

A year later, he became a member of the commission managing the foundation of the University of Berlin. Schleiermacher married Henriette v. Willich, who had been widowed by the early death of his friend. In 1810, W. von Humboldt invited him to participate in the Scientific Deputation on the Committee for Public Schooling, and Schleiermacher worked on carrying out Stein's reforms. That same year, at the newly founded University of Berlin, he became the first dean of the Faculty of Theology. In 1813, he participated in the "Landsturm", which mobilized the population against the French. He founded and edited a journal, the "Preußischer Korrespondent", and was accused of high treason due to his courageous opposition to reactionary movements. For the rest of his life, Schleiermacher was observed by the police and under attack from reactionary political groups. In 1815, he was removed from the Committee for Public Schooling. Two years later, his lectures on the theory of state were censored. In 1819, Schleiermacher vigorously opposed state interventions in university affairs and supported Professor de Wette. De Wette had lost his job as a consequence of the turmoil created by the politically motivated murder of the dramatist August von Kotzebue. In 1823, he was interrogated by the police under the accusations of having offended the crown and of being involved in the gymnastics movement. At the time, gymnastics possessed a political dimension. During the Napoleonic occupation of Germany, Friedrich Ludwig Jahn conceived the idea of restoring the spirits of his countrymen by developing their physical and moral powers through the practice of gymnastics. Jahn opened the first open-air gymnasium in Berlin in 1811. The movement spread rapidly and, after the Napoleonic regime had ended, it was watched closely by the authorities.

Schleiermacher publicly demanded a free press and democratic rights for the universities. The authorities also reacted aggressively to these provocations. In 1824/25, in an ecclesiastical dispute, he spoke against the "despotism of the church". After many years of hard work, he died on 12 February 1834, with many of his scientific manuscripts unfinished.

These quite formal facts sufficiently illustrate Schleiermacher's life, but they tell us little about his general importance. We will therefore highlight two aspects which will serve to explain the development of Schleiermacher's worldview and political positions: 1. Schleiermacher's struggle for a philosophical and political position during his time at the Charité hospital in Berlin (1796 – 1802). 2. Schleiermacher's path towards political maturity and the Napoleonic occupation. These two aspects will help us understand the social and intellectual conditions under which Schleiermacher's views developed. We can only understand his lectures on dialectics if we also take this context into account. We will be in the position to judge the greatness and the limitations of his role in history.

Schleiermacher's struggle for a philosophical and political position during his time at the Charité hospital in Berlin (1796 – 1802)

In 1796, Schleiermacher accepted a position as a chaplain at the Charité hospital in Berlin. A year later, he became close friends with Friedrich Schlegel and became involved with the early Romantic movement in Berlin. In earlier years, he had already read Kant, Leibniz, Shaftesbury, Spinoza, Aristotle, Plato, Rousseau and Montaigne. Now he was looking for philosophical conversations in the cosmopolitan company of the Berlin Romantics. [146] This close contact lasted for five years, until Schleiermacher abruptly severed his ties and became a preacher in Stolp in the summer of 1802. During the Berlin period, Schleiermacher wrote two treatises, which are usually considered his most important works: *On Religion* (1799) and the *Soliloquies* (1800). Both were written under the sway of early Romanticism.

The young movement of Romanticism was inhomogeneous and full of contradictions, and Schleiermacher's relationship to the Romantics was a special one. For one thing, most of the early Romantics were writers with no fixed income.[147] Schleiermacher's professional situation was more secure. He was able to help out Friedrich Schlegel, who was often short of money and trying in vain to obtain a position at the university. In this point, Schleiermacher's financial situation differed from the situation most early Romantics found themselves in.

Like the early Romantics[148], Schleiermacher had hailed the French Revolution enthusiastically. In 1791, he had declared his sympathy for that "good nation"[149], the French, and in 1793 he had written in a letter to his father that "all in all, I love the French Revolution very much. Of course, … I cannot approve of the consequences of human zeal and exaltation."[150] But Schleiermacher rejected the revolutionary terror in France, which most early Romantics did not, and brought forward the humanistic position "that politics can never serve as a justification for murder"[151]. Therefore he also hoped that Germany "may never fall for the harmful fantasy of imitating it [the French Revolution – H.-J. P.]"[152]. Here, Schleiermacher was closer to the bourgeois and nationalist tendencies in German political life, which hoped that reforms could achieve in Germany what the Revolution had done for France.

Schleiermacher's way of seeing Antiquity and the Middle Ages also deviated from the Romantic mainstream. Early in life, he began reading Plato, Aristotle, Lucian, and others. In Berlin, Friedrich Schlegel, who was constantly planning new projects, urged him to collaborate on a translation of Plato. But the two never got far in their joint project because Friedrich Schlegel never stuck to long-term plans and discovered Catholicism through Indian philosophy. Eventually, Schlegel joined the conservative rollback.[153] But Schleiermacher, for his entire life, remained true to the ideals he had found in Antiquity. From 1802 until 1809, he worked on the five volumes of his translation of Plato, which would prove to be extremely influential in 19th century Germany.[154] When, at a later point in time, he had become a professor at the University of Berlin, he gave lectures on Greek philosophy. Schleiermacher profoundly believed in Protestantism and was a great admirer of the Reformation. The upswing of interest in the Middle Ages, which was inspired by Catholicism, left him cold. As opposed to his Romanticist friends, Schleiermacher did not end up rejecting dialectics[155], but committed himself to reinterpreting and reconstructing dialectics, drawing on Plato. His lectures on dialectics were the consequence of his attempts to apply the method of dialectics to science and epistemology. He began lecturing on this topic in 1811, relying on ideas which he had already developed in 1803 in his *Foundations of a Criticism of Traditional Moral Doctrine*.[156]

The question of subjective idealism created another divide between Schleiermacher and the early Romantics. Even though Schleiermacher had been influenced by the Romantics and therefore tended to glorify inner values and creative individuality, and even though he was deeply impressed by Fichte's "I", he always tried to maintain the connection to reality. He did not share the early Romantics' specifically aesthetic perspective on subjective idealism, saw no direct link between art and religion and, generally speaking, kept his distance from Romantic poetics.

"But when somebody separates philosophy and life as strictly as Fichte does", he wrote to Brinckmann, "why should I be so impressed? He must be a great, one-sided virtuoso, but he lacks integrity as a human being."[157] Fichte, Schleiermacher complained, paid too little attention to human individuality.

Much to the annoyance of his Romanticist friends, Schleiermacher was a preacher. Nevertheless, he opposed contemporary Protestant theology. As early as 1789, he sent the following lines concerning the Christian dogma to Brinckmann. Some of the statements from the letter would reappear in his 1799 treatise *On Religion*: "Without it [dogma – H.-J. P.], Christianity would probably never have become what it is today. It could have been so useful that it might never have caused any harm."[158] Nevertheless, in 1790, Schleiermacher passed his theological examination in Berlin.

Schleiermacher had spent his youth in the very religious surroundings of the Moravian Church, an experience which had made a lasting impression on him. Influenced by ideas of Rousseau and Herder, he developed an ethical theory stressing the importance of individuality. Schleiermacher's religious views, which rely on feeling and intuition, arose from these influences. He rejected abstract rationalism and the uninspired, petit-bourgeois belief in human reason, two tendencies which loomed large in Germany during the time of the French Revolution. Sharing this project with young Hegel, Schleiermacher attempted to use religion to transpose the achievements of the French Revolution to Germany. Just like Hegel[159], he had lost interest in the Kantian categorical imperative and was fascinated by mystical, anthropological pantheism and Spinozism.

He hoped to maintain his religion and therefore situated religiosity in the domain of intuition and feeling. Schleiermacher claimed that religion was an "intuition of the infinite"[160], echoing thoughts from the early Enlightenment. Enlightenment thinkers had aimed to make religion "reasonable" by bringing God within the reach of human reason, establishing religion as the ultimate *aim* of moral life and bourgeois society. Schleiermacher, in contrast, found the essence of religion in sentiment and considered it an *impulse* towards morality. As Schleiermacher saw it, religion remained an "intuition of the universe" and the feeling accompanying this intuition[161], but it no longer was a source of ethical and philosophical instruction and was completely detached from ethics and philosophy. This view transformed many religious concepts and it was Schleiermacher's way of supporting an understanding of religion that made room for personal freedom and scientific progress, while at the same time being a liberal manifestation of the bourgeois ideals of freedom, independence and individuality.

These views were first expressed, under the sway of early Romanticism, in the treatise entitled *On Religion*, which must be read together with the *Soliloquies* since some of the thoughts from the latter "clearly predate 'On Religion'"[162]. By publishing these texts, Schleiermacher made an active intervention in the so-called "Controversy of Atheism".[163] Schleiermacher knew from the beginning that these treatises would provoke the accusation of atheism.[164] He claimed that God and immortality were no longer necessary preconditions of religion[165], and he "quite annihilated" the "common idea of religion"[166]. He also claimed that religion, morality and metaphysics had to be kept separate, and that mixing them up meant damaging each individual field. "It [religion – H.-J. P.] is not so ambitious of conquest as to seek to reign in a foreign kingdom"[167] Religion, according to Schleiermacher, was linked to the "feeling for infinite and living nature…, whereof the symbol is variety and individuality"[168]. Religion was to become influential once the sciences and the arts had put a human face on the world. Inspired by a profound feeling

Fig. 45. Schleiermacher in younger years

of optimism concerning the human capacities for recognition and creativity, he concluded that a time would come in which "the perfecting of sciences and arts, those dead forces will be made serviceable to us, and the corporeal world, and everything of the spiritual that can be regulated, be turned into an enchanted castle where the god of the earth [that is, mankind – H.-J. P.] only needs to utter a magic word or press a spring, and what he requires will be done. Then for the first time, every man will be free-born; then every life will be at once practical and contemplative"[169].

Schleiermacher's concept of human liberty is also very important in this context. Like Fichte during his pre-critical period, Schleiermacher believed in mechanical determinism and spoke of an "absolute system of causal relationships in the human world. In Schleiermacher's case, we can even assume that Spinoza was much more important ..."[170]. On the other hand, Schleiermacher was referring to Fichte when he declared human freedom the supreme value: "And so, Freedom, you are for me the soul and principle of all things."[171] Schleiermacher's worldview arose from the synthesis of both philosophical positions:

Human beings grow up under the laws of the universe. The universe shapes individual existence and it cannot escape the influence of the universe. But, according to

Schleiermacher, the universe aims to create "free spiritual beings". Humans can understand this through religion and by gaining awareness of themselves in self-contemplation.[172] Therefore, Schleiermacher saw individuality as a consequence of the laws of the universe.

If a human being succeeded in developing his character, he was free – not a sentient being that had freed itself of the universe (as in Fichte), but a free, moral form of life endowed with free will. By contemplating the universe in religion, one could also contemplate a form of life far superior to one's own. As a consequence, one had to feel the obligation to become a free individual. God was the personification of and finite human concept for this superior form of life, and He "has at least the truth of a poetic symbol of what humanity should be"[173].

Schleiermacher believed that the easiest way for the individual to understand the universe was to understand its spiritual forms of life, human beings. To the religious person, "each man is meant to represent humanity in his own way, combining its elements uniquely"[174]. By interacting, human beings were capable of learning more and more about the universe and of embodying it in their own special way. Therefore, in Schleiermacher's view, human beings were involved in a process of collective learning, striving to perfect their lives as individuals. Everybody had to become a "compendium of humanity"[175]. By gaining awareness of him- or herself, a human being was to become a free and strong character in a morally sound social environment. Therefore, a morally acceptable form of life was only possible in solidarity.

While Fichte's philosophical system had stressed freedom, equality and brotherliness, only freedom and brotherliness remained in Schleiermacher's philosophy. The revolutionary concept of total equality among citizens had been replaced by the post-revolutionary, liberal ideal of perfect individuality. This new ideal reestablished the link to Neo-Humanism and reinterpreted the tradition of thought going back to Herder and Goethe.

But Schleiermacher's subjectivism, which had only grown stronger in the context of Romanticism, remained a secondary factor compared to thoughts about mankind and the unity of mankind: "The more every one approaches the Universe and the more they communicate to one another, the more perfectly they all become one. No one has a consciousness for himself, each has also that of his neighbour. They are no longer men, but mankind also. Going out of themselves and triumphing over themselves…"[176]

In Schleiermacher, radically subjective and idealistic tendencies, which appear especially in the last two parts of the *Soliloquies*, merge with the pessimistic and withdrawn worldview of the early Romantics. But, even though he relegated the dream of a better world to the future and to religious practice, Schleiermacher never lost touch with the real world. Joining the Romantics and Fichte, he became part of the choir of voices bewailing the German misery: "Accept thy harsh lot, O my soul, to have seen the light only in such dark and wretched days. You can hope for nought from such a world to further your aspirations, it offers nothing for your inner development! You will necessarily find association with it a limitation, rather than an enhancement of your powers."[177] "How unavailing is the struggle of a superior mind, seeking moral cultivation and development, with this world that recognizes only legality…"[178] But these statements also show how important the world was for Schleiermacher and that he firmly believed in a brighter future: "O how deeply I despise this generation, which plumes itself more shamelessly than any previous one ever did, which can scarcely endure the belief in a still better future and reviles everyone who dedicates himself thereto…"[179] "… my own belief that I shall meet with nothing that can hinder the progress of my self-development or drive me from the goal of my endeavors, lives in me because of past acts."[180] Schleiermacher's moral worldview rested upon faith, and action motivated by faith, because – as he saw it – the ideals of "eternal" reason had failed in the French Revolution and in bourgeois everyday life. This was a combination of acquiescence and a hesitant but optimistic attempt to change the world. Schleiermacher believed that all human endeavors should strive to improve morality and society. The church, the state and marriage were to be the pillars of this project. But since Schleiermacher saw no chance of establishing this moral community in late 18th century Germany, he could only experience it in a circle of close friends. He said of himself: "I am a prophet-citizen of a later world, drawn thither by a vital imagination and strong faith; to it belong my every word and deed."[181] It was this profound feeling of trust in human progress which motivated him in later years to fight for bourgeois reforms despite the violent resistance of the feudal rollback.

Therefore it comes as no surprise that Emil Fuchs, one of the most important religious socialists of the Weimar Republic, discovered Marx by reading Friedrich Schleiermacher.[182]

Schleiermacher's path towards political maturity – hesitation and commitment

Disagreements quickly dominated Schleiermacher's relationship to Friedrich Schlegel, the most important representative of early Romanticism in Berlin. Shortly after publishing his *Soliloquies*, Schleiermacher explained his new intentions in a letter dated December 1801: "Apart from Plato, we will [walk] *completely different paths* in our literary work. He will present his quite poetical theoretical philosophy by writing poetry, and I will present my *practical philosophy* [emphasis mine – H.-J. P.] in a number of works…"[183] In the following years, Schleiermacher became increasingly critical of Friedrich Schlegel.[184]

Schleiermacher's interest in practical change made him different from the Romantics of 1802. His rugged practical-mindedness, similar to Fichte's temperament, resurfaced during his stint in Stolp. He was not interested in theoretical systems with no connection to reality. His aim was to actively help the human being in gaining awareness for his or her responsibility and personality: "One doesn't have to worry", he wrote in a letter from 1803, "whether or not a treatise which is not a purely ideal construction, but which aims to influence the way people think and live, will be completed. Such a treatise will either be published when it is needed and helpful, or it will never be written in the first place."[185] Isolated from scientific circles in his exile as a minister in Stolp (Pomerania), which began in 1802, Schleiermacher began to work on a book bringing forward a critique of moral theory by critically reviewing Kant, Fichte, Spinoza, Plato and Helvetius. While early Romanticism had emphasized the individual dimension of the self-aggrandizing Fichtean "I", Schleiermacher began to rediscover the importance of a free human society. In the summer of 1803, after having lived through a profound personal crisis, he had fallen in love with Eleonore Grunow, the wife of a colleague he was friendly with in Berlin, Schleiermacher tried to get back on his feet.[186] In November of 1803, he kept a cool distance from the "poetical school" of early Romanticism.[187] He started leaning towards Schelling. The "individuation of the identical", which to him expressed a basic fact about the world, turned into a core element of his theory of morality and dialectics. He began to regain his epistemological optimism: "All life is continuous becoming", he wrote in a letter to Henriette Herz. "Life cannot stand still, it constantly moves ahead in an uninterrupted path of development."[188]

During his days in Stolp, Schleiermacher kept working on the issues that were bothering him: how to become an individual in a social community. He criticized those "obviously infamous" clerics who would stick to their jobs "as long as their parish dishes out 1000 taler" and accused them of being "completely unaware of higher spirituality"

Schleiermacher on philosophical systems (Schelling and Fichte)

"But certainly individuality is the best point of departure because it contains, simultaneously, the aspects of generality and identity. What else could the world be if not the individuation of the identical? And is it possible to grasp this when one only views things from one single perspective, as Schelling does, despite his famous indifference? … I always feel suspicious when somebody constructs a system by using just one single point of departure. When, like Fichte, somebody wants to create knowledge merely for the sake of constructing a dialectical argument, he will get knowledge, and nothing but knowledge. … The same thing happened to Schelling when he tried to understand nature. Now, if somebody had really grasped nature! But whoever wants nature to be so and so to begin with, will hardly come close."[189] (Letter to Brinckmann, 14 December 1803) and of representing "a primitive and worldly way of thinking".[190] He raged against the German Philistines, who "usually find nothing in the heavens, because they have no imagination ... All they are interested in is reason, reason in the bourgeois world, that is, the only world they know ..."[191]

To him, preaching was the last option for changing human nature under the given circumstances: "Today, preaching is the only way to influence the public opinion of the masses on a personal level. Naturally, from a realistic point of view, the image it offers to the people is a bleak one, for it doesn't do much to change the world. But if there is somebody who speaks in public, who takes matters seriously and talks about how things should be, and not about how they are, and if we could be led to believe that only one or two people are paying attention, that must still be quite gratifying."[192]

In the following years, Schleiermacher always strove to bring preaching and lecturing together. In 1805, in a letter from the town of Halle, he wrote: "I certainly expect to be able to show the students how speculation and faith come together by showing them how preaching and lecturing are connected; and I hope to enlighten and move them in two different ways."[193] Preaching was his way of appealing to the listener's emotions and character. Lecturing at the university remained confined to the realm of reasoning.

Schleiermacher's work at the University of Halle fell into a time of transition. We will have to take note of his friendship to Henrik Steffens, whom he considered "a true priest of nature".[194] Steffens encouraged Schleiermacher to take a closer look at Schelling's philosophy of nature: "I take him to be the school's deepest thinker, and a thinker who approaches philosophical problems from more than one angle. In this sense, I even prefer him to Schelling"[195], he wrote in a letter to G. von Brinckmann in December of 1804.

Schleiermacher continued to work on his practical philosophy and gave lectures on philosophical ethics, into which he introduced thoughts from the philosophy of nature and early Idealism. The creation of a truly humanistic world remained something to be expected from the future. The Prussian army's defeat against the French in the battle at Jena and Auerstädt in October of 1806 was an event that led Schleiermacher, like many other bourgeois patriots, to let go of intellectual daydreams and focus actively on the problems which history had put on the agenda.

Schleiermacher considered Napoleon a "tyrant"[196]. The only good thing he saw in Napoleon's war against Germany was the fact that it might lead to an "inner renewal" of the German nation. To him, Napoleon was the destroyer who through his destructive force "helped" activate progressive tendencies in Germany. As early as June 1806, Schleiermacher mused that a "fundamental battle" was inevitable: it would be a battle "for our convictions, our religion, our spirit, as well as for our everyday freedom and material possessions. ... [A] battle ... which the nations will fight alongside their kings..."[197] But Schleiermacher also expressed thoughts which were critical of the present and which

Fig. 46. Henrik Steffens (1773 – 1845)

placed hope in the future. Referring to Germany, he wrote: "Generally speaking, the political situation has so far been an empty promise and mere illusion. The gulf between the individual and the state and between the educated and the masses is far too wide for state and masses to mean something. This illusion must disappear, and only on its ruins will truth be able to arise. A general renewal has become inevitable …"[198]

Seeing himself as a martyr, Schleiermacher stood at the lectern and appealed to his listeners to resist the French oppressors, trying to improve the self-confidence of the lower and middle classes. Schleiermacher's political maturity arose from these political interventions. He took a stand in the raging "battle of political factions in Germany". His formerly suppressed urge for active participation was unfolding rapidly, and Schleiermacher completely lost interest in the contemplative and withdrawn worldview of early Romanticism. In autumn 1807, he commented on this development: "Those isolated, small aspects of life have become completely insignificant in the grand scheme of things. If only I could contribute a tiny element to the big picture, this would now make me happier than being a big fish in a small pond."[199] In the meanwhile, his hopes for a new Germany took on a more radical form: "In order to create a new Germany, the old Germany will probably have to be destroyed even more."[200] Nevertheless, his demands were never as radical as Fichte's democratic political vision.

A large majority of the Romantics had turned towards Catholicism. Schleiermacher, for his part, had turned his back on the Romantics: "Literature is almost dead", he told his boyhood friend G. von Brinckmann in July 1812. "I despise those who become catholic out of sheer weakness ..."[201].

Collaborating with Freiherr von Stein, Wilhelm von Humboldt, Ernst Moritz Arndt, and others, Schleiermacher had become part of the small group of bourgeois intellectuals actively involved in Prussian political history.[202] He played a part in three reform projects: He worked on a church reform, prepared reform projects in the university world and, as a member of the council for culture and education, he also was involved in the measures that were being taken to improve the school system. Shortly before the founding of Berlin University, Schleiermacher took advantage of the occasion and published his personal views on the question (Schleiermacher 1808).

When Wilhelm von Humboldt, who in 1809 was the managing director of the council for culture and education and therefore dealing with the process of founding Berlin University, read Schleiemacher's treatise, he immediately released all other advisors from their duties and relied exclusively on Schleiermacher. Three fundamental political tendencies structured Schleiermacher's views on the question: "Firstly, motivated by true patriotism, Schleiermacher defended the nation's cultural and scientific tradition in his treatise ... Secondly, he opted for independent scientific institutions and independent administrative procedures, unimpeded by the interests of the state or the church. And thirdly, he demanded democratic structures for the highest institutions of education."[203]

But, as early as 1809, Schleiermacher felt that his hopes for a new German nation were in vain. German patriots had hoped to organize a public uprising. Schleiermacher had been actively involved in the preparations[204], but historical conditions were not right and the German monarchs had withdrawn their support. Schleiermacher had established a union of active patriots, the "Tugendbund", but Frederick Wilhelm III declared the project illegal and it disappeared from the scene. When Schill, the commander of a Berlin army regiment, attempted to spark a national revolt, the result was insignificant.

These events shook Schleiermacher's belief in his ideals: "To me, our Prussia still seems like a floating island", he wrote in a letter to Brinckmann in December 1809, "which could go under as easily as it could stabilize itself. My hope in a realistic renewal of the state, which began so well, is waning. And since our achievements are being undermined, sooner or later things might suddenly collapse."[205]

In 1813, Schleiermacher was particularly busy organizing armed militias. He was convinced that the presence of these groups in the entire country would guarantee national unity. Working as a journalist for the "Preußischer Korrespondent", the patriots' newly founded voice, he protested vehemently against the humiliating armistice with Napoleon (4 June 1813). As a consequence, the political forces of the feudal rollback declared him a key enemy, and Schleiermacher was spied upon until his death.

He hoped that war would break out again and smother the reactionary and undecided political forces in the Prussian government, while "bringing the true voices of public opinion to power and giving the nation true pride, which still has not arisen from the anonymous masses"[206].

Only one year later, Schleiermacher was in a position of political isolation and disillusioned about his ideals for the German nation.[207] He felt as though his views made him "a stranger to most people" and that his contemporaries had no "awareness for the spirit and the needs of our time". He also complained about a general lack of insight into "the creative force of history" and that "wherever I turn, the future is becoming bleak".[208] In the following years, the authorities gradually removed Schleiermacher from all influential positions, treating him like most progressive reformers and attempting to neutralize his political authority. But Schleiermacher stuck to his liberal views and the bourgeois reform projects dating back to the War of Liberation against Napoleon. Fearlessly he defended these reforms against the political forces attempting to erode them.

On a number of occasions he intervened in the political battle for citizens' rights and bourgeois influence. When the dramatist Kotzebue was murdered by a student, the authorities used this as an excuse to crack down on the patriotic movement. A colleague of Schleiermacher, Professor de Wette, had gotten into trouble over the affair. Schleiermacher declared his solidarity with de Wette and considered "illegal" means in the fight for the gymnastics movement and the independence of the universities.

Schleiermacher's path towards a mature political position not only made him an enemy of the "higher church authorities", but, as he put it, of "literally the entire ruling class".[209] Schleiermacher's friendships were also affected by his political agenda: "Nothing has remained from the old days except my heart"[210], he told his old friend Brinckmann in 1822. Now, Neo-Humanistic thinkers and liberal-minded bourgeois scientists and politicians such as Alexander and Wilhelm von Humboldt, E. M. Arndt and Georg Reimer were on his side.

In September 1832, one and a half years before his death, Schleiermacher mournfully summed up the political state of affairs: "Today, on the street, I had a long talk with Alexander von Humboldt, a fervent liberal who is very angry about the current state of German politics. I don't feel exactly the same, but I am certainly not happy about how things are going … Often, it makes me sad to think that, when I leave this world, it will be my fate to see our German homeland in such a difficult condition, after having had so many plans and hopes …"[211]

To be sure, the long-lasting hostility between Hegel and Schleiermacher is a curious historical phenomenon: While Hegel, the greatest theoretician of the revolutionary emancipation of the bourgeoisie, supported reactionary tendencies in real life, Schleiermacher, who took up a rather a-historical position in his philosophical views, was an enthusiastic and charismatic fighter for the new German middle class in its struggle for political power.

Schleiermacher's lectures on dialectics

Schleiermacher held his first lectures on dialectics in 1811. He would keep lecturing on dialectics for over twenty years. He was constantly trying to make his thoughts on the subject clearer and more comprehensive. Therefore, we know of important revisions of the lectures, dating from 1814, 1818, 1822, 1828 and 1831. Schleiermacher did not live to see the publication of his *Dialectic*, a project very dear to him. Schleiermacher's *Dialectic* was first published in 1839 by Jonas, who could rely on original notes. This edition was centered on the series of lectures Schleiermacher had held in 1814, and it featured remarks and drafts from other years in the appendix. But this edition is hard to understand for the reader. As the Schleiermacher specialist Wehrung argued convincingly, the different developmental stages of the text – which also were subject to a number of external influences – do not form an organic and logical body of text, but represent separate stages of Schleiermacher's struggle for clarity. This also makes it hard to say which of Schleiermacher's drafts especially inspired the Graßmann brothers, given the fact that they only had access to this compilation edited by Jonas.

An analysis of the history and genesis of Schleiermacher's *Dialectic* and its concepts is beyond the scope of the present biography. Therefore, only a few essential tendencies and aspects of *Dialectic* will be mentioned here. They will give us an idea about what Schleiermacher was aiming for and how influential his *Dialectic* was.

Schleiermacher's interest in dialectics was connected to the project of creating a university for Berlin. He was in favor of strengthening the sciences during the Napoleonic occupation of the country. Schleiermacher was convinced that "structural changes in education" would effectively "annihilate the national spirit", and therefore he did his best to contribute to the founding of Berlin University as a "shelter for German arts and sciences"[212]. In Schleiermacher's way of conceptualizing human values, knowledge had become as important as faith. In his *Occasional Thoughts on Universities in the German Sense* ("Gelegentliche Gedanken über Universitäten in deutschem Sinne", 1808), he took up the Enlightenment's ethical perspective on science and placed it in the context of the fight for a unified, bourgeois nation state. He was very clear on the political dimension of science in bourgeois emancipation: "...the more the spirit of science is active the more the spirit of freedom is active as well ..."[213]. Discernment, Schleiermacher believed, frees "the person from servitude to any authority ..."[214]

Schleiermacher's philosophical disagreement with Fichte and, to a lesser extent, with Schelling was the decisive impulse prompting him to create his own version of dialectics. In *Occasional Thoughts* he remarked that "the entire university, however, the prospering of its entire enterprise, rests on its *not being the empty form of speculation* [emphasis mine – H.-J. P.], wherewith its young charges are alone satisfied. It rests, rather, on the fact that from the immediate perspective available in reason and its activity, insight develops in terms of the necessity and compass of all real knowing, so that from the very outset any presumed opposition between reason and experience, speculation and empirical base is abolished."[215] These views were strongly influenced by Schleiermacher's prior interest in the philosophy of nature and the rapidly evolving natural sciences. Henrik Steffens, a natural scientist and philosopher of nature, was a friend of Schleiermacher in Halle and played an important part in this context. Schleiermacher's resistance against "pure speculation" in philosophy, against the way philosophy ignored empirical experience, made him focus on the problem of knowledge, as Kant had before him. His philosophy and the process of solving this problem were one. Therefore, Schleiermacher saw himself as an epistemologist and theoretician of science (in a wider sense). On the one hand, his philosophy had to deal with the structure of all knowledge. On the other hand, it had to investigate the question what knowledge was in the first place.[216] Apart from these two aspects, the third element of philosophy was the "true dialectical art"[217], which, according to Schleiermacher, was a way of unveiling knowledge and which he later explained in the "technical part" of his *Dialectic*.

Schleiermacher's *Dialectic* is part of the heritage of classical bourgeois philosophy, and it was an expression of the general struggle to find a theoretical answer to the awe-inspiring progress in the natural sciences. *Dialectic* was a clear political statement in a time of "true political battles", battles in which bourgeois reformers tried to combat the outdated social conditions of the country. This context gave Schleiermacher's *Dialectic* its progressive tone. Like Kant's *Critique of Pure Reason*, it aimed to unite empiricism and rationalism. Of course, Schleiermacher relied on the dialectical thinking of contemporary philosophical developments. But Schleiermacher's project failed: he was incapable of finding a coherent solution to this enormous problem, which drove him to revise his system constantly. This situation made his *Dialectic* an intermezzo in the history of philosophy. In the moment of its completion, the political implications of Hegelian philosophy and the influence of Kantian philosophy on the natural sciences, which still had no inkling of developmental thinking, unfortunately made the philosophy of Schleiermacher seem second rate.

Schleiermacher drew inspiration from a large number of philosophical systems. He relied on Plato, Spinoza, Herder, Kant, Fichte and Schelling, integrating their thoughts into a new structure. But many of these elements were incompatible, and they constantly generated new contradictions in Schleiermacher's attempts to create a synthesis. Schleiermacher was incapable of designing a coherent philosophical *system*. The large

Friedrich Engels on Friedrich Schleiermacher (1839)

"...he was a great man, and I only know one man now living who has equal intelligence, equal power and equal courage – and that is David Friedrich Strauss."[218]

number of drafts produced by Schleiermacher was an expression of his constant struggle with the form and content of his philosophy. But the general goal of his *Dialectic* always remained the same: He was attacking speculative philosophy – even though Schleiermacher himself failed to find a true alternative – and attempting to draw the line between science and religion. "He is continuously fighting the self-isolating tendencies in philosophy: at first, isolation from humanity, from life itself, from everyday labour, from poetry in the original sense of the word … Later [in his *Dialectic* – H.-J. P.] … isolation from real science", Wehrung emphasized in his analysis of the location of the *Dialectic* in Schleiermacher's entire oeuvre.[219]

Schleiermacher's *Dialectic* sought an answer to the question: How is it possible to obtain objective knowledge? Therefore, it remained for the most part in the field of subjective dialectics. Schleiermacher gave different answers to the question concerning the nature of dialectics. On the one hand, he defined dialectics as the collection of "principles of the art of doing philosophy"[220], on the other hand he considered dialectics an "organon of knowledge, that is, the repository of all forms of constructing knowledge"[221]. The concept of dialectics dating back to Antiquity represented a third dimension of Schleiermacher's philosophy. In this context, he defined dialectics as "a transformation of thought into language, which guarantees complete mutual understanding by always striving for the highest level of perfection, the idea of knowledge"[222]. This is to say that Schleiermacher's fundamental aim was not to create an abstract and theoretical system, but to find a useable methodology for discovering knowledge. Therefore, Schleiermacher's theoretical reflections on the circumstances under which knowledge may arise and on the nature of knowledge were always directly linked to the epistemological question which procedures and rules were capable of establishing something as knowledge. "When one wants to grasp the rules of conjunction in a scientific way," Schleiermacher emphasized, "then they cannot be separated from the deepest foundations of knowledge. For in order to conjoin correctly, one cannot deviate from the way in which things are really conjoined, and the relationship between our knowledge and the things is our only proof."[223]

Schleiermacher's philosophical approach was tightly linked to his critical reception of Fichte's subjective idealism: "A purely positive I and a purely negative non-I generate no world whatsoever."[224] "And if the organic activity is related to the activity of reason: then we create the organic impressions ourselves."[225] The underlying problem of separating "knowledge from unspecific, arbitrary ways of thinking"[226] could only be solved when the content of thinking was posited as being independent of thinking. These were the initial principles which took Schleiermacher back to Kant. Schleiermacher reestablished the separation of rational understanding ("Verstand") and sense-oriented impressions ("Sinnlichkeit") in thinking. In different contexts, Schleiermacher called these two elements reason and organization, intellectual function and organic function, or reason

Schleiermacher on thinking, being and knowing

"In every thought something that is being thought is posited outside thought." [227]

Knowledge "is that kind of thinking which corresponds to an existent."[228]

Knowledge is "the correspondence of thought to an existent".[229]

"Every thought that refers to something that is posited outside it, but does not correspond to the thing that is posited outside it, is not knowledge."[230]

and sense-oriented impression. According to Schleiermacher, thinking was the combination of reason and organization. Reason was to fulfill a guiding function, organization an animating function – reason gave form, and organization, sensual material. But more than Kant, Schleiermacher believed in the necessity of "organic affection". To Schleiermacher, the external world was more than a "thing in itself". It was a totality, a reality full of oppositions. Therefore, the criterion for truth had to be the correspondence of thinking and being.

Wehrung, whose perspective was informed by philosophy of immanence, therefore felt obliged to say the following about Schleiermacher's *Dialectic*: "The firm belief in the importance of sense-perception, along with reason, returned as the firm belief in the reality of an existent which had to give content to thinking and which made knowledge definite."[231]

Schleiermacher's concepts of space and time also expressed the gnostic and historical distance between Kant and himself. To Kant, space and time were *nothing but* forms of sensual intuition.[232] Schleiermacher, in contrast, objectively attributed them to external entities. Every objective being which existed in a world of oppositions inevitably had to exist in space and time.

Schleiermacher did not merely postulate that it was possible for categories as different as thinking and being to come together. He derived this insight from internal empirical experience in self-consciousness. According to Schleiermacher, it was in self-

The modern aspects of Schleiermacher's concept of space

"Space and time belong to the object and are not confined to our representation of it."[233]
There is no such thing as empty space, "for if it is not filled with matter, it is filled with action. … Space is the unfolding of being, time the unfolding of action. Internal space or filled space is a representation of a thing's diversity within the unity of its existence. Every filled space is a representation of an internal opposition. Where this opposition ceases to exist … spatial relations also cease to exist."[234]

consciousness that human beings experienced immediate identity, the immediate correspondence of their own existence and thinking. From this type of correspondence, Schleiermacher extrapolated the possibility and necessity of the correspondence of external existents and thinking, which represented the basis for true knowledge: "Somebody might say that the correspondence of thinking and existents is an empty thought because of the absolute distinctness and incommensurability of both categories. But in self-consciousness we experience both, thinking and the thing we are thinking of, and our life consists of the correspondence of the two."[235]

This seemingly materialistic thesis quickly transformed into a perspective linked to Schelling's philosophy of identity. The identity of the ideal and the real, of ideal and material aspects of human existence led Schleiermacher to establish the identity of the ideal and the real as an inherent characteristic of the "ultimate ground of all being". Generally speaking, Schelling's and Schleiermacher's views concerning the identity of material and ideal aspects in absolute being were the same. But the two philosophers differed in their epistemological point of departure. Schelling's experience of identity arose from intellectual intuition,[236] Schleiermacher's from inner experience. Obviously, Schelling's *Lectures on the Method of Academic Study* ("Vorlesungen über die Methode des akademischen Studiums") were very influential here, especially lecture number one.[237] In Schleiermacher, the correspondence of the ideal and the real did not remain restricted to the external world, to nature. He also applied this principle to thinking and thereby introduced further changes to epistemology. Reason and organization, and therefore also concept and "schema", are the equivalents of the ideal and the real in exterior being.

"The activity of reason is grounded in the ideal, but the organic activity is dependent upon the effects of objects in reality: this is how being is posited both in the realm of the ideal and the real, and the ideal and the real coexist as *modi* of being."[238] Spinoza is a strong undercurrent in this view of the double nature of being. At the same time, Schleiermacher was thinking of Plato when he stated that the intellectual system of concepts was timeless. This also was the case when he claimed that there had to be a natural correlative to the hierarchy of forces and phenomena in being (the string of concepts leading from the general to the particular).[239] While, on the one hand, he kept the ideal world strictly separated from the real world, on the other hand Schleiermacher emphasized that, in the process of *becoming*, real knowledge could only be based upon the identity of both sides, that is, the intellectual and the organic functions. Concerning the relationship between thinking and existents, the tension between the material and the ideal was one of difference and identity, and Schleiermacher's responses to this tension differed greatly in his drafts of *Dialectic*. But he also developed impressive dialectical thoughts, such as his reflections on the unity of thinking and language or the interrelations of reason and perception in recognition.

Schleiermacher on thinking, language and speech

"Language only comes into existence through thinking and vice versa. They both need the other to become complete."[240]

"…all speech is a manifestation of thinking. Therefore, [thinking] … is a kind of *inner intellectual activity which only becomes complete through speaking*."[241]

"*what does it mean to think* … Answer: *It is the kind of intellectual activity which becomes complete by becoming identical to speech, and which refers to something which is posited as being external to this activity.*"[242]

Speech "is a manifestation of existents, on the one hand, and on the other it is a conjunctive structure of thinking. A word and segments of speech must have the same double nature. … Every word symbolizes existents and the conjunctive structure of thinking at the same time. … Words are material elements of speech, but also always conjunctive signs, or formal elements of speech, and also always manifestations of a part of being."[243]

As for Schelling, the realization that reality was a totality of interconnected oppositions was fundamental for Schleiermacher. Therefore, as he saw it, the act of progressing in oppositions was one of the basic functions of recognition. Schleiermacher had begun by deriving the identity of thinking and existents, of the ideal and the real in absolute being from the feeling of self-consciousness, and he had continued by thinking about the infinite "universe". This led him to attempt to better determine absolute being conceptually, which to him meant speaking about God. On his path from specific conceptual oppositions to more general oppositions, the highest opposition of the real and the ideal confronted Schleiermacher with a boundary. This opposition on the highest level of generality represented, according to Schleiermacher, the limit under which "the system of oppositions extends itself"[244]. Schleiermacher believed that the idea of being as the *absolute* unity of the real and the ideal was "no longer a concept"[245] of understanding. "Concerning its content, this idea is a concept, but concerning its form, it is not."[246] This was Schleiermacher's answer to the weaknesses of Schelling's concept of the absolute. Absolute identity, without the slightest opposition, could not be a point of departure for the project of understanding reality and its oppositions from a theoretical point of view. Absolute identity provided no reason for opposing tendencies to arise.[247] This prompted Schleiermacher to say that the idea of the absolute unity of being, from the formal point of view, was not a concept: "for nothing can be said about it. It is nothing but an act of positing, but it cannot diversify itself, it may not be structured as a system of oppositions"[248]. The transcendent foundation of knowledge – Schleiermacher used Kantian terminology – was therefore unattainable. It lay beyond the realm of concepts. Nevertheless, Schleiermacher did not give up on the idea of an absolute unity of the ideal and the real, because doing so would have meant the end for the concept of God, which stood for this unity. But these theoretical constructions turned the concept of God into an empty indifferent formula, and Wehrung was correct when he noted that, in Schleiermacher's *Dialectic*, "God was not identical to the limits [of concepts and judgments – H.-J. P.] which Schleiermacher had found, but beyond them. There was no way of proving His existence; Schleiermacher had to limit himself to offering an explanation: if there was a God, He could not be characterized by worldly and sensual predicates. For human beings, divinity had to be a 'nothing'[249] … that is, a non-being, inexpressible. Every attempt at picturing the divine meant placing it in an inadequate context."[250] This was the path which Schleiermacher had already begun to take in *On Religion*, namely the project of criticizing religion in order to renew it. His compromises and contradictions were "corrected" at a later point in time by left-wing Hegelians – such as Strauß and Feuerbach – or by anti-rationalist thinkers such as Schopenhauer and Nietzsche.

Schleiermacher on the aim of dialectics

The aim of dialectics is "…to discover the inherent existence of God as the foundational principle of all knowledge, while considering this principle only a way of constructing true knowledge."[251]

By attempting to define and classify the main scientific fields from the perspective of this transcendent foundation, the notion of God, Schleiermacher was forced to establish a "notion of the world" along with his concept of God. In Schleiermacher's view, God and the world were not identical, but contrasting notions: "In our thoughts, God is always posited as unity, and not as multiplicity, while the world is multiplicity without unity."[252] Nevertheless, he went on to say, we cannot really "posit multiplicity without … linking it to a certain form of unity"[253]. He reduced the force of this opposition by describing the world as a unity, considering it a "totality of oppositions".[254] Schleiermacher was clearly attempting to avoid an openly pantheistic worldview, and the following quotation shows us how far he was willing to go: "God = a unity which excludes all oppositions; World = a unity which includes all oppositions."[255]

Even though God and world were supposed to be transcendent concepts, their transcendent qualities were incomparably distinct. According to Schleiermacher, human knowledge could continuously come closer to grasping the world. God, in contrast, was inaccessible to human thinking. Only emotions were capable of opening this door.[256] Concerning the discovery of new knowledge, Schleiermacher's God represented the fundamental range of possibilities (a theoretically constitutive principle), while the world represented the norm, a way of measuring the objective value of knowledge (a practically constitutive principle).[257] This amounted to saying that the concept of God was irrelevant to the actual process of generating new knowledge. This led Hermann Graßmann's brother Robert to conclude that Schleiermacher, as a thinker, was in fact an atheist: "He says nothing about the existence of God beyond the world. There is no way God could have existed without the world, and therefore we learn nothing about His existence before our world existed as well. The events depend on God, which is to say: they are linked to the laws of nature, and therefore divine intervention in miracles is unthinkable. As a thinker he can leave God out, and his system remains the same; ..."[258]

The world was the only point of reference for Schleiermacher's system of scientific fields. It was the true origin of knowledge. This meant that Schleiermacher's views often connected Spinozism and pantheism to objective idealism, which gave his *Dialectic* its peculiar double nature. Its structural support was Schleiermacher's dialectical method: " ...he [Schleiermacher – H.-J. P.] is better at distinguishing things clearly, at drawing lines, contrasting and conjoining concepts dialectically, than at putting together single particulars to form a whole and developing the individual moments of an idea organically", E. Zeller wrote about Schleiermacher. "He loved to use a given as his point of departure, to distinguish its different elements, to view it from contrasting perspectives, using one perspective to deal with the other, one aspect to explain the other, and vice versa. This is how, gradually, after thoroughly exploring all aspects of an argument, he finally came to a definite conclusion."[259] But Schleiermacher's dialectical philosophy had the same limitations as Schelling's: it was not a *developmental dialectic*. Even though Schleiermacher viewed the world as a totality of oppositions, he never grasped the development of these oppositions. Despite the fact that he defined the unity of these oppositions as a unity of interrelated effects, as a "chiasma" of oppositions, this unity was never negated and taken to a higher level. A simple and inferior opposition could never generate a new, superior opposition (the famous Hegelian "Aufhebung"). In Schleiermacher, when the nature of oppositions changed, this always was a change in the sense of "more" or "less", but never a movement which generated new oppositions. In this respect, Schleiermacher was never as profound a thinker as Hegel. Schleiermacher's "structural and functional dialectic" was certainly capable of stimulating the development of the natural sciences at the time – in fact, even more so than Hegelian dialectics. But it offered no model for dynamic developments in society. Therefore, Schleiermacher's *Dialectic* was an adequate expression of his strictly liberal, but anti-revolutionary political views. It comes as no surprise that Schleiermacher's political position included the demand for constitutional monarchy. Evolution was already a part of his *Dialectic*, but revolution was not.

In the second, "formal or technical part" of his *Dialectic*, which represents about ⅖ of the entire work, Schleiermacher investigated "knowledge in motion, in the process

P. Ruben on Schelling's dialectic of nature

P. Ruben's words also apply to Schleiermacher's *Dialectic*:
"Schelling's concept of the opposition as dualistically counterpoised forces or processes is an expression of the fact that he did not grasp the real unity of natural objects properly, which is to say that he did not grasp the dialectical opposition properly. In his view of the scientific understanding of nature, dualistically opposed processes can only exist when the (macroscopic) system in question is a closed system and when its external parameters always remain the same. But this excludes development, and equilibrium is the basis and the final goal of the system."[260]

of becoming". Even though on the whole Schleiermacher's *Dialectic* aroused very little interest in philosophy, this part is usually ignored completely or dealt with in a few remarks.

But it represents – and this is also how Schleiermacher saw it – the true core of his *Dialectic*. It is the focal point of all previous philosophical efforts, and it serves to secure objective knowledge in the process of recognition. At the beginning of the "technical part", Schleiermacher explained once more what dialectics was all about, confirming our view: "Dealing with basic knowledge and rules of combination, it is supposed to be a repository and a criterion. Basic knowledge is made up of two ideas (God and the world). We must develop the rules of combination from these two ideas."[261] Schleiermacher is now using the epistemological groundwork of his *Dialectic* to "construct" methodological tools for different scientific fields.

Schleiermacher emphasized the importance of induction, which was to form the basis for every deduction, and demanded that theory and empirics should form a unity. Wehrung pointed out that "[i]t is quite remarkable that, at the time, Schleiermacher … admitted that the deductive work of the natural sciences was strongly influenced by induction, that they were correct to rely on experience!"[262] Schleiermacher was very much aware "that all speculative theories about nature are very shaky, and how much knowledge has been gained about nature by relying on the opposite method."[263] Schleiermacher's attempt to establish practical knowledge as a guideline for bourgeois science was a highly relevant contribution to the German bourgeois mindset shortly before the Industrial Revolution. We also should not overlook the influence of bourgeois Humanists such as Wilhelm and Alexander von Humboldt, who were friends of Schleiermacher, on his final concept of scientific inquiry in separate disciplines. But deduction was overvalued in his reflections, and Schleiermacher was incapable of amending this problem. Schleiermacher kept returning to deduction. Wehrung characterized Schleiermacher's dilemma correctly when he wrote: "*His motto was: we must get as close as we can to reality; it was not: we must take reality as a point of departure.*"[264]

Principles from Robert Graßmann's *Theory of Scientific Discovery* which relate to Schleiermacher's *Dialectic*:

"(1201): The human mind generates concepts. All concepts generated by the mind in one given field form a unity.

(1202): All concepts generated by the mind in one given field must essentially be one, while at the same time every single concept must be essentially distinct from all others.

(1203): The human mind posits specific qualities by which it determines concepts, separates one from the other and distinguishes them.

(1204): Every specific quality and distinctness is the consequence of the act of positing an opposition. Every opposition consists of (no more than) two concepts. The first concept is the contrasting term to the second, and the second the contrasting term to the first.

(1205): A concept and its contrasting term are generated from a unity by an opposition. The mind posits oppositions within unities.

(1206): Therefore, concept and contrasting term form an opposition within unity. This means that every one of the contrasting terms constitutes the unity only in a one-sided way.

(1207): A unity can be transformed into two contrasting terms when the two sides of two oppositions respectively are present in every one of the two oppositions, but connected in the first, and opposed in the second, in short, connected crosswise. This amounts to saying that the two contrasting terms in the unity are generated by a chiasma of two oppositions, by connecting two oppositions crosswise."[265]

In the technical part of his *Dialectic*, Schleiermacher attempted to explain his "heuristic method", which was supposed to enable scientists to find true knowledge in a process of recognition. He underlined the importance of analogy and congruence, of observation and experimentation. He rejected the way Idealism constructed reality in speculative thinking. The technical part of *Dialectic* expressed the book's central message: The importance of leaving the dichotomy of philosophy and empirical scientific disciplines behind!

The Graßmann brothers were especially impressed by the second part of the second section of the "formal or technical part" of Schleiermacher's *Dialectic* – paragraphs 330 through 346. In this part, Schleiermacher dealt with the question how new knowledge could be attained and how this knowledge could be integrated into a scientific context. This was highly relevant for Hermann Graßmann, who was aiming to found a new branch of science, and for Robert Graßmann as well, who wanted to reconstruct the entire edifice of science. The unity of induction and deduction, of scientific presentiment,

A. Twesten made the following notes after listening to Schleiermacher's first lecture on dialectics in 1811:

"Today Schleiermacher began his long-awaited lectures on dialectics. It turned out to be a wonderful event. He demonstrated how, when philosophy is separated from the real sciences, or, as in past historical epochs, when philosophy contradicts them, it can only become an unhealthy and crippled science..."[266]

analogy and proof, architecture and heuristics of knowledge, which, if one wanted to construct it correctly, had to form a double relative opposition – all of these elements aroused their interest. They were explicitly discussed in Robert's *Edifice of Knowledge* and brilliantly put to the test in Hermann's *Extension Theory*.

Everything Schleiermacher had written before reaching the "formal or technical part" of his *Dialectic* had served to create the foundation for real and potential knowledge, thereby establishing the preconditions of a methodology for attaining true knowledge. His aim had been to support the rational elements of Schelling's philosophy of nature with a synthesis of philosophical views from Kant, Spinoza and Plato. This philosophical structure had been constructed to comply with the requirements of the natural sciences.

Schleiermacher's *Dialectic* was a magnificent attempt to break through the boundaries of bourgeois classical philosophy in Germany. But, due to Schleiermacher's ambition of creating a philosophy of religion, it had to fail. Almost immediately, it became a historical footnote and it left almost no traces in public philosophical debates.

Notes

1 This textbook would never have been published "had it not been the will of the highest authority to use one single textbook for the entire mathematical curriculum in all secondary schools. Therefore, we either needed to put together such a book, or we needed to choose among the ones we already had." (1835, p. iv) Given the fact that this textbook contains, more than any prior work, interesting thoughts on the negative magnitude in mathematics and on the sum and the product in geometry, phenomena which Hermann Graßmann intensely worked on in 1830/31 and 1834, it is quite surprising that M. Radu's (2000) remarkable analysis ignores the textbook completely.

2 M. Radu links Bartholdy (and, as a consequence, J. Graßmann) only to Humboldt's secondary-school reforms, in which Bartholdy played a significant part (Radu 2000, p. 87sq.). But concerning their two books, this connection was irrelevant because they had

been written with the schools for the poor in mind. Also, it is risky to connect J. Graßmann to Kant and Fichte via Humboldt (ibidem, p. 88sq.) because, in fact, Bartholdy and Schleiermacher were close friends. Therefore, since Bartholdy was a follower of Pestalozzi, his theories on individuality, intuition and construction (as in the case of J. Graßmann) probably were inspired by the latter. This fact reduces the force of all arguments claiming that Graßmann was inspired by I. Kant, J. Schultz and J. F. Fries.

3 J. Graßmann 1824, p. iv.

4 It is surprising that in her excellent analysis of the contemporary concept of mathematics, M. Radu (2000) mentioned neither J. Schmid, nor Pestalozzi. Many of Graßmann's ideas were taken from J. Schmid and were not originally his own.

5 J. Graßmann 1817, p. 1.

6 Ibidem, p. 7.

7 Ibidem, p. 6.

8 See Diesterweg 1956, p. 195 – 197, 264, 525, and Diesterweg 1959, p. 224.

9 See Schmid 1809, p. XIXsq.

10 Schmid 1809, p. xv.

11 Ibidem, p. xx.

12 See ibidem, p. xxii.

13 Ibidem, p. xxiii.

14 Ibidem, p. xxiv.

15 Ibidem, p. xix.

16 Ibidem, p. xxv.

17 See, in this context, Diesterweg 1956, p. 195.

18 The introduction of the concept of movement into geometry during the first decades of the 19th century is usually attributed to methodological considerations of d'Alembert. See Molodschi 1977, p. 176.

19 J. Graßmann 1817, p. 18.

20 Ibidem, p. 57.

21 Ibidem, p. 1.

22 Cantor 1879, p. 598.

23 J. Graßmann 1824, p. 3sqq.

24 Concerning Pestalozzi's and Schmid's concept of intuition, see Spranger 1959, Memmert 1963, de Moor 1999.

25 See Kant 2007, p. 70.

26 We can find the beginnings of a theory of combinations in Pascal (1665). Leibniz supposedly coined the term "ars combinatoria" to designate the theory of combinations. Wallis and Jacob Bernoulli I went on to elaborate this theory. See Leibniz 1960, p. 495, editor's note.

27 See KRY, p. 175.

28 Schmid 1809, p. 122.
29 Ibidem, p. 124sq.
30 Leibniz 1960, p. 84/85.
31 J. Graßmann 1817, p. x/xi.
32 In the further course of the present book, the quotations from *On the Concept and Extent of the Pure Theory of Number* have been taken from an unpublished translation of this work by Lloyd Kannenberg. The page numbers, however, refer to the German original.
33 ZL, p. 1.
34 Ibidem.
35 Ibidem.
36 ZL, p. 2.
37 Ley 1969, p. 182.
38 See chapter 4, section 2.
39 ZL, p. 3.
40 ZL, p. 3. See, in this context, I. Kant in Kant 2007, p. 52/53, p. 100/101. Robert Graßmann's rejection of Kant is obvious. See R. Graßmann 1890b, p. 68 – 73. Here we read: "All in all, Kant worked with great diligence and sharp-mindedness, but he remained stuck in pure subjectivity, a state of mind which is incapable of understanding the outside world and our life in the world, apart from also being incapable of gaining awareness for God's revelation in the world." (ibidem, p. 73)
41 ZL, p. 4.
42 In *On the Concept and Extent of the Pure Theory of Number*, J. Graßmann remarked in a footnote that he was currently working on a treatise "on general and geometric combination theory" (ZL, p. 2).
43 ZL, p. 5.
44 See chapter 4, section 3.
45 ZL, p. 9/10.
46 Ibidem.
47 Ibidem, p. 10.
48 Ibidem.
49 Ibidem. For the relevance of this view today, see Kummer 1972, p. 284.
50 ZL, p. 11.
51 Ibidem, p. 11/12.
52 Ibidem, p. 12.
53 ZL, p. 13.
54 See ZL, p. 14.
55 See ZL. p. 17sq.
56 Lorenzen 1969, p. 91.
57 ZL, p. 21.

58 J. Graßmann 1835, p. 10. See also KRY, p. 194/195.

59 Ibidem, p. 26sqq.

60 Ibidem, p. 18. See also Heuser 1996. But Heuser's view that organic development and the history of nature stand for a concept of "superiority" is questionable. See Heuser 1996, p. 50.

61 ZL, p. 39.

62 Ibidem, p. 40.

63 KRY, p. vi/vii.

64 Concerning J. Graßmann's mathematical treatment of crystalline structures, see Quenstedt 1873, p. 58sqq, and the remark by F. Engel in GW22, p. 244sqq. Hermann Graßmann presented an elegant approach to the same theory (1839), directly continuing his father's work.

65 KRY, p. vii.

66 KRY, p. ix.

67 KRY, p. 184.

68 KRY, p. 174. This has a very modern ring to it. See Poincaré 1906, p. 19 – 21.

69 KRY, p. 172.

70 KRY, p. xii/xiii.

71 KRY, p. 173.

72 Ibidem, p. 134.

73 Ibidem, p. ix.

74 Ibidem, p. 12.

75 Ibidem.

76 See Scholz 1996, p. 40sq.

77 Ibidem.

78 See chapter 3, section 3.

79 See KRY, p. 34/35.

80 See chapter 3, section 3.

81 Radu 2000, p. 95.

82 ZL, p. 7.

83 Ibidem, p. 171/172.

84 KRY, p. 166.

85 KRY, xvii/xviii.

86 Letter from Hermann Graßmann to his brother Robert, 3 Mai 1835. Quoted from BIO, p. 57.

87 Letter from Robert Graßmann to Friedrich Engel, 10 March 1896. Quoted from BIO, p. 133.

88 For this information and in the following lines, we can rely on Schubring 1996d and R. Graßmann 1890a. G. Schubring must be credited for tracking down Robert Graßmann's

secondary-school diploma, his curriculum vitae and examinations in the archive of the University of Greifswald. See Schubring 1996d. During the translation of the present book, the author finally managed to discover a portrait of Robert Graßmann in the Szczecin archives.

89 See in this context: R. Graßmann 1890a, p. XIXsqq.

90 See the following document from the archive of Greifswald University: Universitätsarchiv Greifswald, UAG Wiss. Prüf. Komm. Nr. 144/38.

91 Graßmann's diploma, issued 31 December 1831. Quoted from BIO, p. 41.

92 See Hermann Graßmann's curriculum vitae of 1833. Quoted from BIO, p. 21.

93 According to Robert Graßmann. – See BIO, p. 91/92.

94 See Hermann Graßmann's Foreword to the *Extension Theory* of 1844 (A1, p. 16).

95 R. Graßmann 1875a, p. 116/117.

96 R. Graßmann 1890a, p. xx.

97 A1, p. 16.

98 Apart from Robert Graßmann, the later minister of defence, G. von Kameke, a teacher from the "Friedrich-Wilhelmschule" (Jungklaß) and Graßmann's brother-in-law Scheibert, the headmaster of the "Friedrich-Wilhelmschule", belonged to this circle. See BIO, p. 92.

99 Gert Schubring (1996d) answers all these questions affirmatively.

100 R. Graßmann 1890a, p. xx.

101 R. Graßmann 1890b, p. 11.

102 Ibidem, p. 82.

103 Ibidem, p. 82/83.

104 Ibidem, p. 83.

105 R. Graßmann 1890c, p. 312 – 314.

106 J. Graßmann 1835, p. iv.

107 Letter from C. G. Scheibert to Ludwig Wiese, 1 May 1861. In: Schulze 1906, p. 91.

108 R. Graßmann 1890c, p. 312/313.

109 Ibidem, p. vii.

110 See Birjukov 1960.

111 See Hoffmann 1966.

112 Concerning M. Ohm, see Eikenjäger 1977, p. 35.

113 See chapter 3, section 6.

114 Friedrich Engels: Ernst Moritz Arndt (1841). In: MECW, vol. 2, p. 141.

115 R. Graßmann 1875a, p. 80/81.

116 Ibidem, p. 82.

117 R. Graßmann 1875a, p. 82.

118 Friedrich Engels: The Constitutional Question in Germany. MECW, Volume 6, p. 75sqq.

119 R. Graßmann 1876a, p. 246.

120 R. Graßmann 1890e, p. 174/175.

121 Despite his rejection of Hegel, Robert Graßmann relied on some elements of Hegelian dialectics when he developed his *Theory of Scientific Discovery* ("Erspähungslehre", R. Graßmann 1890c, p. 509sqq.). Of course, he ignored the Hegelian concept of negation. – See R. Graßmann 1890c, p. 511sqq.
122 R. Graßmann 1890e, p. 174/175.
123 See section 3 of the present chapter.
124 R. Graßmann 1875a, p. 20.
125 R. Graßmann 1890e, p. 268.
126 R. Graßmann 1876b, p. 123.
127 R. Graßmann 1890b, p. xxx/xxxi.
128 R. Graßmann 1876b, p. 131.
129 See Kuntze 1909b.
130 See ibidem, p. 280sqq.
131 Ibidem, p. 284/285.
132 R. Graßmann 1875b, p. 63.
133 Ibidem, p. 66.
134 R. Graßmann 1875a, p. 10.
135 Fischer 2001, p. 137.
136 Ibidem.
137 Ibidem, p. 75.
138 R. Graßmann 1890b,p. 83.
139 Ibidem, p. 84.
140 BIO, p. 21/22.
141 See chapter 2, section 2.
142 Fischer 2001, p. 13.
143 Ibidem, p. 12.
144 See chapter 1, footnote 51.
145 Meisner 1922, p. 16.
146 See also Schleiermacher's letter to his sister Charlotte, 27 September 1797. In: Meisner 1922, p. 94.
147 See in this context Heinrich 1976, p. 62sq.
148 See in this context Heinrich 1976, p. 37/47.
149 Letter from Schleiermacher to Sam. Henri Catel, 24 May 1792. In: Meisner 1922, p. 66.
150 Letter from Schleiermacher to his father, 10 February 1793. In: Meisner 1922, p. 73.
151 Ibidem.
152 Ibidem.
153 See Geerdts 1966, p. 280/281.
154 See Wirzberger 1973, p. 78.
155 See in this context Heinrich 1976, p. 79sq and 199sq.

156 See in this context Wehrung 1920, p. 12sqq.
157 Letter to G. v. Brinckmann, 14 December 1803. In: Meisner 1922, p. 332. See also the letter to G. v. Brinckmann of 4 January 1800. In: Meisner 1922, p. 163.
158 Letter from Schleiermacher to G. von Brinckmann, 28 September 1789. In: Meisner 1922, p. 48.
159 See Ueberweg 1923, p. 82.
160 Schleiermacher 1893, p. 278.
161 See Schleiermacher 1893, ibidem.
162 Fuchs 1969, p. 35.
163 See ibidem, p. 31.
164 In June of 1801, Schleiermacher wrote a letter to his theological superior F. S. G. Sack, in which he defended himself against the accusation of being a follower of Spinoza. In: Meisner 1922, p. 212.
165 See Schleiermacher 1893, p. 9.
166 Schleiermacher 1893, p. 276.
167 Ibidem, p. 20.
168 Ibidem, p. 278.
169 Schleiermacher 1893, p. 178/179.
170 Fuchs 1969, p. 46.
171 Schleiermacher 1926, p. 18.
172 "Within me I can behold nought but Freedom", Schleiermacher wrote in the *Soliloquies* (1926, p. 18).
173 Schleiermacher 1926, p. 24.
174 Ibidem, p. 31.
175 Schleiermacher 1893, p. 79.
176 Schleiermacher 1893, p. 180.
177 Schleiermacher 1926, p. 53.
178 Ibidem, p. 60.
179 Ibidem, p. 50/51.
180 Ibidem, p. 73.
181 Ibidem, p. 62.
182 See Fuchs 1969, p. 133.
183 Letter from Schleiermacher to E. von Willich, 13 December 1801. In: Schleiermacher 1914b, p. 28.
184 See also a letter to E. Grunow, 10 September 1802. In: Meisner 1922, p. 270/271.
185 Letter from Schleiermacher to A. von Dohna, 10 April 1803. In: Meisner 1922, p. 301.
186 See the letter from Schleiermacher to E. von Willich, 10 August 1803. In: Schleiermacher 1914b, p. 69.

187 See Schleiermacher's letter to G. v. Brinckmann, 26 November 1803. In: Meisner 1922, p. 322.
188 Schleiermacher's letter to H. Herz, 17 December 1803. In: Meisner 1922, p. 331/332.
189 Letter from Schleiermacher to G. von Brinckmann, 14 December 1803. In: Meisner 1922, p. 331.
190 Letter from Schleiermacher to E. Grunow, 8 July 1802. In: Meisner 1922, p. 250.
191 Letter from Schleiermacher to E. Grunow, 12 August 1802. In: Meisner 1922, p. 255.
192 Letter from Schleiermacher to H. Herz, 16 September 1802. In: Meisner 1922, p. 275.
193 Letter from Schleiermacher to E. and H. Willich, 26 November 1805. In: Schleiermacher 1914b, p. 141.
194 Letter from Schleiermacher to H. Herz, 27 March 1805. In: Meisner 1923, p. 33.
195 Letter from Schleiermacher to G. von Brinckmann, 15 December 1804. In: Meisner 1923, p. 29.
196 Letter from Schleiermacher to Ch. von Kathen, September 1806. In: Meisner 1923, p. 67.
197 Letter from Schleiermacher to Ch. von Kathen, 20 June 1806. In: Ibidem, p. 64.
198 Letter from Schleiermacher to G. Reimer, November 1806. In: Ibidem, p. 72.
199 Letter from Schleiermacher to Ch. von Kathen, autumn of 1807. In: Ibidem, p. 95.
200 Letter from Schleiermacher to Friedrich von Reimer, 12 January 1807. In: Ibidem, p. 87.
201 Letter from Schleiermacher to G. von Brinckmann, July 1812. In: Ibidem, p. 148.
202 See Dilthey 1985.
203 Schuffenhauer 1956, p. 42.
204 See, for more information, Schuffenhauer 1956, p. 44sqq.
205 Letter from Schleiermacher to G. von Brinckmann, December 1809. In: Meisner 1923, p. 122.
206 Letter from Schleiermacher to the countess von Voss, 7 June 1813. In: Ibidem, p. 185.
207 See Schleiermacher's letter to countess von Voss, January 1814. In: Ibidem, p. 201.
208 Letter from Schleiermacher to countess von Voss, 7 January 1814. In: Ibidem, p. 209.
209 Letter from Schleiermacher to G. von Brinckmann, 19 February 1822. In: Ibidem, p. 322.
210 Letter from Schleiermacher to G. von Brinckmann, 19 February 1822. In: Ibidem, p. 323.
211 Letter from Schleiermacher to G. von Brinckmann, September 1832. In: Ibidem, p. 364/365.
212 Letter from Schleiermacher to G. von Brinckmann, September 1811. In: Meisner 1923, p. 138.
213 Schleiermacher 1991, p. 52.
214 Ibidem, p. 51.
215 Ibidem, p. 27.
216 See Schleiermacher's explanations in Schleiermacher 1991, p. 17.
217 See ibidem, p. 29 and 35sqq.

218 Friedrich Engels: Letter to Friedrich Graeber in Berlin, 12 July 1839. MECW, vol. 2, p. 457.
219 Wehrung 1920, p. 75.
220 DIAL, p. 8.
221 Ibidem, p. 22.
222 Ibidem, DIAL, p. 261.
223 Ibidem, p. 7.
224 Ibidem, p. 129.
225 Ibidem, p. 77.
226 Ibidem, p. 95.
227 DIAL, p. 48.
228 Ibidem, p. 48.
229 Ibidem, p. 50.
230 Ibidem.
231 Wehrung 1920p. 109.
232 See Kant 2007, p. 348.
233 DIAL, p. 335/336.
234 DIAL, p. 336.
235 DIAL, p. 53.
236 See Schelling 1859, p. 255.
237 Schelling 1859, p. 215sqq.
238 DIAL, p. 75/76.
239 See Schleiermacher's words in DIAL, p. 102sqq.
240 DIAL, p. 261.
241 Schleiermacher 1942, p. 126.
242 Ibidem, p. 125.
243 DIAL, p. 249.
244 DIAL, p. 77.
245 Ibidem, p. 86.
246 Ibidem.
247 See also Ruben 1975, p. 37.
248 DIAL, p. 86.
249 Here Wehrung is referring to Schleiermacher in DIAL, p. 416.
250 Wehrung 1920, p. 306.
251 DIAL, p. 17.
252 DIAL, p. 162.
253 Ibidem, p. 165.
254 Ibidem, p. 162.
255 Schleiermacher 1942, p. 303.

256 See Wehrung 1920, p. 155.

257 See DIAL, p. 164/165.

258 R. Graßmann 1890b, p. 83/84.

259 Zeller 1875, p. 609.

260 Ruben 1975, p. 36/37. In the 1830s and 40s, Schleiermacher's and Schelling's concepts of dialectics were influential in the conservative Romantic political theory in Germany. They also turned into the theoretical basis for the uncompromising position of the liberal bourgeoisie during the 1848 Revolution.

261 DIAL, p. 173.

262 Wehrung 1920, p. 289.

263 DIAL, p. 559.

264 Wehrung 1920, p. 289.

265 R. Graßmann 1890c, p. 517 – 520.

266 According to Henrici 1889, p. 177.

3 Hermann Günther Graßmann's contributions to the development of mathematics and their place in the history of mathematics

> "The history of science … aims to trace the pattern of thought which developed in a community over the generations and to outline the general processes in which the discovery of the particular was more a symptom than a driving force. In adopting such a view, one will seldom come to say that a discovery was made before its time, or that a single personality left the imprint of his mind on his epoch. Instead, the totality of the sciences takes on an organic form. In the individual case, though, it still remains to be seen to what extent almost identical phenomena had causal repercussions upon one another; but one must not confuse chronology and a causal effect."[1] A. Clebsch

In order to do justice to the demands voiced by the exceptional German mathematician Alfred Clebsch, one will have to grasp the significance and the fate of Hermann Günther Graßmann's mathematical oeuvre against the backdrop of the revolutionary changes that occurred in geometric ideas in the first half of the 19th century.

Only in the context of the history of mathematics will it become clear that Graßmann's geometrical-algebraic inquiries and the main tendencies of geometrical research of his time arose from a common mathematical problem, despite the relative distance between Graßmann's theoretical approaches and the specific content of contemporary geometrical research.

The rich inspiration which mathematicians such as Klein, Peano, Whitehead and others received from Graßmann's work even after its significance had been acknowledged in retrospect can be seen as proof of the extraordinary fruitfulness of his mathematical ideas.

3.1 Some basic aspects concerning the development of geometry from the 17th to the 19th century

Starting with the 17th century, the contrast between the state of development of geometry and the extraordinary progress in analysis and algebra had become increasingly clear. While, as early as the 16th and 17th centuries, algebra had introduced calculation with letters and analysis had elaborated functional thinking and infinitesimal methods, thereby replacing ancient standpoints by completely novel approaches[2], Euclid's relatively independent, 2000 year-old geometrical structure had yet remained unshaken. Only with Descartes' (1637) creation of analytical geometry[3] had ancient geometry been combined with the modern algebraic-analytic elements. The ensuing elaboration of the "purely technical" dimension of Cartesian analytical geometry by Fermat (1679, posthumous), Wallis (1655), de Witt (1659), Newton (1703) and Euler (1748)[4] stretched from the late 17th century into most of the 18th century.[5]

But the obvious contrast between the state of development of geometry and that of analysis and algebra had not been in any way resolved. Rather, it had become more pronounced. Often, the approach of connecting algebra, analysis and geometry through the Cartesian coordinate system had not helped to resolve the problems posed by "perception-based" mathematics, represented by mechanics[6]. As early as 1686, a clearheaded thinker like Leibniz had seen this problem. "... but one should know", he noted, "that algebra, the analysis of Viéta and Descartes, is primarily an analysis of numbers, and not of lines, although geometry is indirectly brought back (to arithmetic), given the fact that all magnitudes can be expressed by numbers. But this often forces us to take great digressions; and it is often the case that geometers can prove in a few words what in a calculation is a long procedure. And if one has found an equation in some difficult problem, it is still a long way to finding the problem's structure, which one was looking for. Moving from algebra to geometry is a sure path, but not the best; ..."[7] And even 200 years later, Engel had reason for similar reflections.[8]

Tightly linked to the search for an adequate mathematical approach to problems in physics, this "limitation" of Cartesian geometry stimulated the creation of modified and extended notions of coordinates. The work done by Möbius (1827), Graßmann (A1) and Plücker is representative of this effort.[9] Analogous reflections led, for the first time with Leibniz' ideas relating to a purely "geometrical characteristic" (1679), to the search for a way of dealing directly and algebraically with geometrical objects.[10] It would take another 150 years before Graßmann (1832/1840) obtained mathematically relevant results in the Leibnizian line of thought. Starting from the same basic outlook, Graßmann created the foundations for vector- and tensor calculus.

Taking place in an independent line of inquiry, investigations on the geometrical interpretation of complex and hypercomplex numbers and on the mathematical representation of torsions and rotations in space – Wessel (1797), Gauß (1799), Argand (1806), Hamilton (1843)[11] – also led to the establishment of vector algebra.

Yet another mathematical contradiction concerning the traditional concept of dimensionality arose from the interplay of analytical and geometrical methods in analytical geometry. While three-dimensional space had so far gone unquestioned in geometry, there now was no justification for this limitation in analytical calculations with generalized coordinates, especially concerning their applications in mechanics.[12] This is why we find attempts to go beyond the traditional empirical notion of dimensionality in geometry mostly in thinkers such as the analytical geometers Wallis (1685), d'Alembert (1764) and Lagrange (1780).[13] Finally, we find explicit n-dimensional algebra in Cayley[14] (1843), Graßmann (1844) and Schläfli (1850/52). The process of improving the understanding of geometry, which had a large part in the split the concept of space was undergoing, separating into different concepts in mathematics and physics, went along with intense discussions of idealist philosophy, particularly concerning the concept of Kantian apriorism.

The contradictions which had arisen in the evolution of analytical geometry and the search for an answer peaked and came to a preliminary solution in Klein (1872), bringing about a more comprehensive and unified concept of geometry. In his 1871 dissertation, Lie, one of the fathers of geometry's newly found form, remarked in retrospect: "As is well known, the rapid evolution of geometry in our century is closely linked to philosophical considerations on the nature of Cartesian geometry..."[15] But we would only grasp the progress of 18th and 19th century geometry incompletely were we to omit two additional geometrical lines of inquiry, in addition to the process described so far.

One is concerned with attempts to prove the dependency of the axiom of parallels to the rest of Euclid's axioms, a project running all the way back to antiquity. Wallis, Saccheri, Legendre, Lambert, Schweikart and Taurinus came to important results which contributed to the later understanding of the problem.[16] But only at the beginning of the 19th century, when rapidly evolving engineering technology had more and more mathematicians directing their attention towards geometry, did three important mathematicians – Gauß (1817), Janos Bolyai (1831) and Lobačevskij (1826) – almost simultaneously and independently from one another reach the conclusion that a novel, "non-Euclidean geometry", based on the negation of the Euclidean axiom of parallels, was possible.[17]

Riemann's, Beltrami's, Helmholtz's, Klein's and Lie's[18] far-reaching reflections concerning the relation of the concepts of space in physics and in mathematics were prompted by the results these three scientists had obtained by questioning the traditional concept of space, results therefore bearing an enormous philosophical relevance. The effort for leaving behind the ancient concept of geometry had gained momentum.

Apart from the emerging non-Euclidean geometry, we must also emphasize the establishment of projective geometry as an independent mathematical discipline at the beginning of the 19th century. In retrospect, we can also trace certain aspects of this second line of inquiry to Antiquity, notably to Apollonius of Perga and Pappus.[19] The theory of perspective received an important impulse during the Renaissance, when it was elaborated in a practical and artistic sense by craftsmen, artists and fortress architects, such as Alberti, Piero della Francesca, Leonardo da Vinci and Albrecht Dürer. Posterior purely mathematical approaches, which we find in Desargues, Pascal, and a century later in the work of the exceptional Alsatian mathematician Lambert, did not reach a wider public. The actual construction of projective geometry, stimulated by long-term military and economic interests, only began with the work of Gaspard Monge.[20] Influenced by the task of planning fortress constructions, he created with extreme originality a new geometrical discipline, descriptive geometry, which considerably simplified calculations in the context of rampart construction. He created a wide following of students by lecturing at the École Polytechnique in Paris, founded in 1794 as a consequence of the French Revolution and at the time the most advanced school of engineering in the world.[21]

The French school of geometry following in Monge's wake drew differing conclusions from his work and split into a predominantly analytical and a more synthetically oriented group, which more or less opposed the use of methods relying on coordinates in geometry. While Carnot, Brianchon and Poncelet were more drawn to the synthetic contents of Monge's work, Victor Poncelet, brilliantly founding projective geometry as an independent discipline during his time as a Russian prisoner of war, when lacking almost all scientific resources, Gergonne, Lamé and Chasles stood for the analytic treatment of the projective properties of geometrical objects.[22]

Poncelet's main treatise, the *Traité des Propriétés Projectives des Figures* (Poncelet 1822), which in 1822 already included all important concepts of projective geometry, such as the cross-ratio, perspectivity and projectivity, represents a movement leading away from Euclid's geometry while also rendering geometrical terminology more precise and expanding its meaning. By separating metrical properties from geometrical objects, Poncelet questioned the seemingly obvious connection between geometry and metrics, raising it as an epistemological problem. The discovery of the principle of duality rendered the ancient idea of the point as a geometrical first element obsolete, leading to inquiries into incidence-geometry which, in turn, prompted the creation of multiple new "geometries".

By the mid 1820s, Poncelet's ideas had spread to Germany. The synthetic treatment of projective geometry reached a high-point and more or less came to an end in the work done by Steiner (1826, 1832) and von Staudt (1847). Von Staudt's 1847 book *Geometry of Position* ("Geometrie der Lage", Staudt 1847) presented an axiomatic foundation for projective geometry which did not rely on Euclidean geometry.

The analytic treatment of projective geometry had its first German representatives in Möbius (1827) and Plücker (1828, 1831). The two scientists created the analytical instruments needed to deal with the projective properties of geometrical objects, which Poncelet had discovered by following a synthetic approach. Möbius' barycentric coordinates and Plücker's line- and tetrahedral coordinates can be seen as part of the effort to deal adequately with projective geometry in an analytic framework.

H. Wußing on the evolution of geometry:

"Actually, what we witness in geometry during that period is not the realization of one particular trend but the evolution – in seemingly divergent directions – of several implicit tendencies about to break free. Fundamental modes of thought in and about geometry were no longer dictated by habit. The geometry whose content, method, and aims were conceived in terms of the millennial euclidean tradition began to change. Once the idea had been abandoned that geometry is something unique (its contents, methods and goals interpretable only in the ancient tradition stemming from Euclid), concepts such as coordinate, length, parallelism and distance; the habit of regarding the point as the fundamental element of all geometry; and the very view of geometry as the art of measurement, all turned out to be capable of generalization and in need of critical review. The dissolution of a seemingly unavoidable way of thinking had the following effect: Through a dialectical evolution the contradictions inherent in the concept of the unity of geometry produced a wealth of 'geometries' determined only by their attributes – 'noneuclidean', '*n*-dimensional', 'line', 'projective', and so on."[23]

After a long and evenhanded struggle between the analytical and synthetic methods of dealing with projective geometry, the analytical approach gained the upper hand by the mid 19th century because it was closer to the entire structure of mathematics and could therefore draw on a wider range of methods.

Now the dialectical contradictions, which had been part of the evolution of Cartesian geometry, reappeared in the more comprehensive analytical-algebraic work on geometrical relationships. The problem had become even more acute, now that the common Euclidean foundation had been destroyed and a plurality of geometrical approaches had taken its place.

In summary, we can identify at least four factors which, beginning with Descartes' analytical geometry, led to the demise of the limited, ancient concept of geometry:

1 An expansion of the concept of coordinates beyond the Cartesian notion and the search for new algebraic-analytical ways of expressing geometrical relationships;
2 the transition to *n*-dimensional manifolds;
3 the creation of non-Euclidean geometry;
4 the removal of the seemingly absolute connection between geometry and metrics and, as a consequence, the connection between projective and metric geometry.[24]

The unfolding of geometry into seemingly divergent tendencies, which began in the first half of the 19th century, led to the formation of counter-movements and, at the same time, to the search for new, uniform foundations of geometry. The turn towards algebraic-analytical methods developed into the driving force behind geometry's further development. It is important to note that in Möbius (1827) and Graßmann (1844) we can already find methodologically coherent steps towards a new structure for geometry, based on generalized and open approaches. But contemporary mathematics did not react to these new ideas. Only in the 1860s did geometers of the English algebraic school make further important progress in this direction, independently from Möbius and Graßmann. "Initial and significant progress was made by the English school surrounding Boole, Cayley, Sylvester and others, by having recourse to what originally was a number-theoretic discipline, the theory of forms, which then in the theory of invariants led to a first instrument for classifying different geometries."[25]

Plücker's student Felix Klein brought the classification of different geometrical approaches to a peak and to a preliminary solution. In close collaboration with the Norwegian mathematician Sophus Lie, Klein was influenced by Jordan's inquiries into group theory and developed a program bringing forward a classification of geometry based on group theory, thereby going beyond the standpoint held by the theory of invariants. It was published in 1872 under the title *A Comparative Review of Recent Researches in Geometry* ("Vergleichende Betrachtungen über neuere geometrische Forschungen", Klein 1974) and came to be known as the Erlangen Program. It was only now that the importance of the work done by Möbius, Graßmann and Hamilton, dating back 30 to 50 years, was recognized and appreciated.

With Klein's Erlangen Program, the 200 year-old process of dissolution of the ancient concept of geometry had come to a preliminary end. The new geometry had become more abstract, while also including a wider range of mathematical structures. Its state of development had adapted itself to analysis and algebra, thereby making it possible for these three disciplines to interweave closely.[26]

3.2 Graßmann's examination thesis on the theory of tides

Hermann Günther Graßmann's mathematical productivity blossomed in the first half of the 19th century under the influence of the complete revolution in geometry which had begun in the late 18th century. His search for new ways of handling geometrical relations was also influenced by the methodological, conceptual or dialectical, and epistemological contradictions in geometry[27] which had stimulated the whole process of theory building. But, in contrast to the majority of his mathematical contemporaries, Graßmann walked a new path in directly connecting algebra and geometry, a path leading him away from projective geometry and taking him towards a vector algebra approach to the development of affine geometry.

Graßmann drew on his father's work on a new "geometrical theory of combination", which had originated from the influence of Pestalozzi's project for elementary school reform.

We find another non-scientific motivation for changing the ancient concept of geometry in the attempts of Pestalozzi and his colleague Joseph Schmid to make geometry *teachable* in elementary schools by leaving the rigid Euclidean form of presentation behind.

Scientific approaches to affine and projective geometry in the early 19th century.

Projective geometry represented the greatest part of geometric scientific inquiry in the first part of the 19th century. Möbius renewed investigations into affine geometry as late as 1827, going back to ideas of Euler's, without finding acclaim in the scientific community. In the late 1850s in this context, Cayley, referring to the projective measurement that bears his name, still enthusiastically exclaimed:
"Herewith metric geometry becomes a part of projective geometry, and projective geometry is all of geometry."[28]

These projects of Pestalozzi and J. Schmid not only inspired Justus Graßmann, they also sparked Jacob Steiner's interest in geometry.[29]

Apart from combinatorial considerations and those of geometric incidence, Hermann Graßmann found ideas for calculating directly with oriented lines in his father's works. According to Graßmann himself[30], he had already developed vector-addition and the generation of vectors of the second and third order[31] as early as 1832 (or, as he put it: geometric addition and geometric multiplication of displacements). But it was only around 1840 that he became aware of the far-reaching consequences of this innovative algebraic treatment of geometry when a problem of mathematical physics, the theory of low and high tides, took him back to his ideas of 1832.[32]

To Graßmann, the analytic-geometric way in which Laplace (Laplace 1799 – 1827) had dealt with the problem seemed too clumsy and too far-removed from the issue. "Laplace almost never indicates", Graßmann noted on the first pages of his text, "which procedures led him to the results he presents; and above all they are so indirect that one cannot even begin to fathom how he came up with this or that strategy. So if one wanted to limit oneself to following his paths, one would feel like somebody who is taken to some unknown place with his eyes blindfolded, and who – even though he takes every step to his destination himself – wouldn't know where he is once he has arrived."[33]

N. Bowditch on Laplace (he was Laplace's translator into English):
"I never came across one of Laplace's 'Thus it plainly appears' without feeling sure that I had hours of hard work before me to fill up the chasm and find out and show how it plainly appears."[34]

Graßmann explicitly emphasized that it was the conflict arising from the traditional way of treating mechanics with a geometry of coordinates that led him to remembering and expanding the vector algebra work he had already done: "I have abandoned the traditional method because, following this method, the introduction of coordinates forcefully distorts nature. This forceful distortion becomes visible in the fact that the way we develop and elaborate our formulas loses touch with nature's way, not sharpening our view of the concept we were interested in, but drawing our eyes away from it, focusing our attention on the transformation and development of formulas which have nothing to do with our original aim."[35]

Graßmann created many of the essential concepts of vector calculus and of exterior algebra in his exceptional graduation thesis, which he needed in order to be awarded the unconditional permission for teaching physics at higher schools. Just the length of the dissertation, running to 190 printed pages, exceeded normal expectations at the time.[36]

Still limited to three-dimensional space, we find here:

- the addition of vectors and bivectors ("geometrical addition of displacements and spatial areas"), with a special emphasis on the associativity and commutativity of this operation;[37]
- also the geometric construction of the center of gravity of a system of point masses, already found by Möbius in 1827;[38]
- the exterior multiplication of two and three vectors, interpreted geometrically as the signed area of a parallelogram or signed volume of a parallelepiped (with a special emphasis on the change in sign when the factors are reversed!);[39]

- the concept of the scalar product of two vectors ("linear product of two displacements"), obtained by projecting one vector onto the subspace ("straight line") of the other;[40]
- as a third form of the multiplicative conjunction of vectors there is the product of a vector into an exponential- and angle-magnitude, the geometrical substrate of which is the rotation of a displacement around its point of application.[41]

Concerning the last three kinds of multiplication, Graßmann explicitly emphasized distributivity with respect to vector addition and used it to build a methodological connection to the principles of common addition and multiplication (an indirect application of the principle of permanence of the laws of combination). We might regard the following as a typical turn of phrase: "If we want to call *geometrical multiplication* that conjunction which is determined in the same way by geometrical addition as algebraic multiplication is determined by algebraic addition, then we will necessarily have to..."[42]

So we should note that as early as 1840, in his examination thesis on the theory of tides, Graßmann implicitly had already found the concept of the linear operator over V^3 ("geometrical magnitude of dimension zero and of the second order"), representing it in a matrix-form (!) and fully comprehending its meaning: "By multiplying a displacement with a geometrical magnitude of dimension zero and of the second order, that line can be transformed into any other value (and into any other direction and size)."[43] A year later, in a remark on the final draft of his treatise, he called these magnitudes "affinity factors" ("Affinitätsfaktoren").[44]

Considering the richness of conceptual creativity Graßmann displayed in this early treatise – even though external factors determining the writing of this work[45] prevented the systematic and logical elaboration of many of these concepts – one will have to count it as the foundational text of vector algebra.

There were hardly any comparable efforts for constructing vector algebra at the time. In the 17th century, Leibniz had conceived the project of constructing a purely "geometrical analysis", sharing with Graßmann the basic idea of working directly with geometry in an algebraic way, but without attaining significant results.[46] Only much later do we find more inquiries into the geometrical addition of oriented displacements with Möbius in 1827 and 1843 (Möbius 1827, 1887a) and, without any connection to Möbius, with Bellavitis[47] in 1835 (Bellavitis 1835). But still, the results these scientists obtained in the field of vector algebra are inferior to Graßmann's work.

Wessel, Argand and Gauß belong to a second line of development, which also led to the creation of vector calculus. Looking for the geometric interpretation of complex numbers, these scientists also found first conjunctions of vectors. These inquiries reached their highest expression in William Rowan Hamilton's work. Starting with problems in mechanics, this important Irish physicist and mathematician searched for an algebra that was supposed to do for rotations in three-dimensional space what complex numbers could do for rotations on a plane. After many failed attempts, he finally found quaternions in 1843. Overestimating the relevance of these hypercomplex numbers, he dedicated the last twenty-two years of his life to developing the theory of quaternions.[48] Large sections of this theory are dedicated to vector algebra and -analysis of three-dimensional Euclidean space. Without the slightest connection to Graßmann, he discovered vector-addition, vector-multiplication, the scalar product of vectors, and more. These accomplishments made him one of the founding fathers of vector calculus, next to Graßmann.

As in Hamilton's case, vector algebra made an enormous impression on Graßmann. Graßmann was so struck by the elegance and explanatory power of this new way of treating affine and Euclidean geometry that he dedicated his entire life to systematically developing vector algebra.

Grappling with Cartesian geometry, Graßmann had broken through to completely new conceptual constructions in mathematics. Graßmann's algebraic-geometrical approaches take up an exceptional position in the structure of mathematical inquiry in the first half of the 19th century. The reason for this lies in the special circumstances under which Graßmann's mathematical productivity unfolded. Felix Klein remarked in this context: "In spite of all the originality and importance of his work, Grassmann was never a university teacher. Indeed, as a consequence of his peculiar development, rightful recognition was altogether denied him during the main part of his mathematical life. Understandably, Grassmann often complained of this unjust fate; and yet it held certain advantages for him, the consequences of which can be seen in his works and personality. We academics grow in strong competition with each other, like a tree in the middle of the forest which must stay slender and rise above the others simply to exist and conquer its portion of light and air. But he who stands alone, like Grassmann, can grow on all sides, can harmoniously develop and finish his nature and work. Of course such versatility as Grassmann embodied must inevitably be accompanied by a certain amount of dilettantism..."[49]

Graßmann's position as an outsider and his originality are grounded in the late awakening of his mathematical interests. His mindset was shaped by autodidactic studies and a penchant towards philosophy, his connections to his father's work and, last but not least, his relative isolation from the scientific world of his time, caused by his job as a teacher in a secondary school in Stettin.

In Graßmann's subsequent projects, these influences would become increasingly relevant in positive and negative ways.

It is worth noting in this context that Möbius and Steiner, whose mathematical work was much closer to contemporary geometers' efforts than Graßmann's, also occupied similar positions as outsiders. Take Steiner, who only possessed incomplete knowledge of analysis and algebra, and who had only become interested in geometry when he was thirty years old. His unique power of imagination made him the rediscoverer of synthetic geometry, standing absolutely alone by the importance of his achievement and personality. This is how Biermann tells the tale: "Steiner's intention of having the Berlin Academy award a prize every other year to people solving problems from synthetic geometry, 'mainly taking into account his own methods and principles' and financially covered by his estate, soon became impossible to carry out for lack of applicants."[50] Steiner's results may have been generally known; his method remained inaccessible.

In Möbius' case, the analogy with Graßmann is even more obvious. Even though his inquiries into the classification of geometrical relationships were closely linked to the main thrust of 19th century geometry, the overall weight of these far-reaching approaches went unrecognized by his contemporaries. Even Gauß could not appreciate their importance. "Humble in his personal attitude and in the style of his publications, maintaining only indirect contact to the leading centers of mathematical science, August Ferdinand Möbius in his lifetime could not win the full respect and recognition he would have deserved."[51] This is why, in his lifetime, he was recognized mostly for his work in applied mathematics.[52]

3.3 The 1844 *Extension Theory* and Graßmann's theory of algebraic curves

In the years after 1840, Graßmann invested a great deal of energy into developing his ideas. The graduation thesis, the importance of which had not been understood by the examination commission's assessor, Prof. Conrad[53], was only edited in 1911 when Graßmann's collected works were published. Graßmann's feeling that the work's structure was flawed had kept him from publishing it. Only in 1844 did Graßmann present the public with his ideas in a major treatise. The book's title alone, *The Science of the Extensive Magnitude or Extension Theory, a new mathematical discipline,... Part one, Linear Extension Theory* (A1), goes to show how seriously Graßmann took his new mathematical concept. The approaches from 1840 had developed by now, growing into a comprehensive piece of theory. "...accessible only with extreme difficulty, indeed

Alfred Clebsch on Graßmann's achievements

"Unfortunately, the beautiful works of this highly important geometer remain quite unknown; this supposedly is the case mostly because, in Graßmann's mode of presentation, these geometrical results are corollaries to much more general and very abstract problems, which, due to their unusual form, present considerable difficulties for the reader."[55]

… almost unreadable…"[54], this treatise contains nothing less than the construction of *n*-dimensional affine algebra.

Graßmann opens his book with philosophical and methodological reflections. They begin with an analysis how to enlarge the focus of mathematics, inspired by the development of his new theory, and then continue with methodological considerations and the new theory's place in the architecture of mathematics.[56] The mathematical part of the book begins with the conception of the "General Theory of Forms". Starting with a perspective on mathematics as a theory of forms, Graßmann analyses in the most abstract way possible the general structures of concrete "conjunctions of forms".[57] Here, he places special emphasis on "elementary conjunctions", demanding they have module properties, that is, associativity, commutativity and an inverse and neutral element. The so-defined conjunction of the first order, or "formal addition", is then followed up by an investigation of a conjunction of the second order ("formal multiplication"), for which he only requires distributivity with respect to formal addition. Graßmann directly posits the validity of the module properties for formal addition and distributivity for formal multiplication *as the principles for constructing* these conjunctions: "This generally is the way", he wrote, "that initially, that is when no species of conjunction is yet given, such a conjunction of next higher order is defined."[58]

Since Graßmann does not require the forms generated by conjunctions of the second order to be embedded in the fundamental domain, he can use this form of conjunction for the formal generation of new mathematical objects in the further course of the text.

According to Graßmann, formal conjunctions must correspond to a "real concept that expresses the method of generation of the product by the factors"[59], that is, by something concrete. In the case of real conjunctions, further structural characteristics can come into play – as do commutativity and associativity in common multiplication. As a consequence, Graßmann's theory of forms includes the principle of permanence as a special case.

In contrast to Hamilton, Graßmann does not take the concept of the field to be one of the most general concepts, but links it to objects of specialized mathematical disciplines, such as arithmetic of real numbers.

After Graßmann has laid down the foundation for all mathematical disciplines by presenting these uniquely generalized group-theoretical and structural abstractions[60], he starts with the actual presentation of his new mathematical discipline. He treats it in two sections entitled "Extensive magnitudes" and "Elementary magnitudes".[61]

The first section mainly expands and generalizes the notion of "displacement calculus" from the examination thesis, but without making any reference to the inner product. Graßmann now leaves behind geometry, which he does not consider an integral part of pure mathematics[62], and turns to the analysis of a "pure *n*-fold extensive entity of elements". In modern terminology, we would simply speak of the *n*-dimensional affine point space.

Instead of conceptualizing the straight line in a geometrical and mechanical approach as an entity created by the movement of a point, Graßmann begins with the "domain of the first order", defined as "all [displacements] generated by continuation of the same fundamental evolution or its opposite".[63] Following a similarly abstract principle, the concept of the "extensive magnitude of the first order" (a free vector) is generated from the concept of the displacement, and the concept of a "domain of the *n*-th order" (the equivalent of the affine point space) is introduced.

In order to give an impression of Graßmann's conceptual developments, I would like to present the following lengthy quotation, which describes how extensive magnitudes of the first order are generated:

> "The extensive structure will therefore only appear as elementary if the evolutions which the generating element undergoes are equal to each other; so that, if an element *b* is produced by an evolution from an element *a*, both belonging to some given elementary extensive structure, then by an equal evolution an element *c* of the same elementary structure is produced from *b*, and indeed such an equality must continue to hold if *a* and *b* are regarded as infinitesimally separated elements, since this equality should obtain throughout for continuous generation. We call such an evolution, in which there is generated from one element of a continuous form its next adjacent element, a fundamental evolution, and can therefore say: 'The *elementary extensive structure* is that which results from the continuous action of the same fundamental evolution.'
> Now in the same sense in which evolutions are regarded as equal, we can also regard the structures generated thereby as equal, and in this sense, that is, that those generated in the same way by equal evolutions are themselves equal, we call the elementary extensive structure of the first order an *extensive magnitude* or an *extension* of the first order, or a *displacement*. Thus the elementary extensive structure becomes an extensive magnitude if we neglect the elements that it includes and retain only the nature of its generation; and while two extensive structures can only be regarded as equal if they include the same elements, two extensive magnitudes are already equal if they are generated in the same way (that is, by the same evolutions), while not including the same elements."[64]

Even in Graßmann's times these concepts did not comply with the standards imposed by mathematical rigor. His audacious attempt to find a basis for his *Extension Theory* without relying on other mathematical disciplines, especially geometry and analysis, sometimes led to rather imprecise conceptual creations (such as "elementary change of an element", "Grundänderung eines Elementes"). The *Extension Theory* of 1844 is characterized by its semi-philosophical style, unusual for mathematicians, and a radically reduced inventory of formulas. We find the reason for this in Graßmann's intention of presenting all concepts in the most general form they could possibly take on, of highlighting the simple basic ideas and of pushing back less important thoughts and contingencies. F. Engel grasped the double nature of Graßmann's intellectual attitude very well, when he wrote: "Even though the work profits a lot from this intention, which even gives it an exemplary quality in some sections, and even though this was how Graßmann gained extremely important insights into the principles of mathematical science – one might think of, for example, his theory of … conjunctions and of his general concept of multiplication –, it nevertheless cannot be denied that Graßmann, in his search for general truths, sometimes lost his footing."[65]

To be fair with Graßmann, one will have to say that this conceptual vagueness remains limited to a restricted number of elementary concepts and to his unsatisfactory critique of Euclidean geometry. The concept of "fundamental evolution", for example, immediately becomes more precise when it is replaced by the concept of the parallel displacement in *n*-dimensional space. With this in mind, all ensuing constructions which Graßmann elaborates are perfectly rigorous.

A comparison with Riemann is revealing.

Graßmann and Riemann on the possibilities of generating higher dimensional spaces	
Graßmann's conceptual development of an *extended system of higher order* in the *Extension Theory* of 1844: "In order to proceed to the conjunction of dissimilar displacements, I consider next two dissimilar fundamental evolutions, and let the first fundamental evolution (or its opposite) arbitrarily transplant an element, and then let the element thus generated likewise be arbitrarily transplanted by the second method of evolution. In this way I can therefore generate an infinite set of new elements from a single element, and the collection of elements so generated I call a system of *second order*. I then assume a third fundamental evolution…"[66]	*Riemann's* conceptual development of the generation of *manifolds of higher dimensions* in the habilitation lecture of 1854: "In a concept whose various modes of determination form a continuous manifold, if one passes in a definite way from one mode of determination to another, the modes of determination which are traversed constitute a simply extended manifold and its essential mark is this, that in it a continuous progress is possible from any point only in two directions, forward or backward. If now one forms the thought of this manifold again passing over into another entirely different, here again in a definite way, that is, in such a way that every point goes over into a definite point of the other, then will all modes of determination thus obtained form a doubly extended manifold. In a similar way one obtains a triply extended manifold…"[67]

If one compares Graßmann's explanations to Riemann's, it becomes clear that both share the same level of philosophical abstraction and the same underlying principle. The only difference between the two is that Riemann, when compared to Graßmann, describes the generation of manifolds in a more general way, going far beyond the framework of affine geometry.

There is no proof for a possible influence on Riemann from Graßmann's *Extension Theory*, published ten years prior to the lecture Riemann held in order to be awarded the title of professor. In the introduction to his lecture, Riemann only points to works by Gauß and Herbart, which he claims to have consulted in preparing the lecture.[68]

But let us return to the mathematical developments of *Extension Theory*. Thanks to his special method of generating an "extended domain of the n-th order" (n-dimensional affine point-space), Graßmann now possesses a system of n pairwise disjunct subspaces, in each one of which we have linearly independent vectors. Expanding on the concept of formal addition, which he had developed in the general theory of forms, and referring to a concept of superimposed "methods of evolution of an extended system's elements" (parallel displacements of points), Graßmann gradually introduces vector addition. "If two displacements are given", Graßmann remarked in this context, "and one evolves an arbitrary element by the part of the first, and then (progressively) by the corresponding part of the second, the collection of elements so generated comprises the sum of those two displacements."[69] Again, this form of putting it must have been rather unappealing to contemporary mathematicians.

Graßmann's intention of disconnecting his inquiries from the specific representation of vectors by basis (in Graßmann's words: "specific method of generation of elements") finds its expression in the conceptual creation of the replacement theorem of basis vectors – completely formalized by Graßmann in 1861 and rediscovered half a century later by Steinitz.[70] He presented a proof for the following formulation of the theorem: "First I will show that if the system is generated by any m methods of evolution whatever, I can replace any given one of them by a new method of evolution (p) belonging to the same system of m-th order and independent of the remaining $(m - 1)$, and generate the given system using this in combination with the other $(m - 1)$."[71]

Also, the concept of linear independence of vectors brought Graßmann to determine the "extended system's" dimensions, when he wrote: "Every system of m-th order can be regarded as generated by those same m independent methods of evolution from any arbitrary element; that is, all other elements can be generated from a single such element by those methods of evolution."[72]

Once Graßmann has arrived at this point in his development, he returns to his initial objective of criticizing the traditional concept of geometry. In accord with his view of geometry as the science of real space, Graßmann only allows those principles in geometry that arise from the "perception of space".[73]

Influenced by kinematic considerations, this basic thought leads him to reduce the geometrical basic theorems to two, which are meant to express the "fundamental properties of space as they are initially imparted to our imagination, that is, its simplicity and relative limitation."[74] According to Graßmann, the simplicity of space is mirrored by the principle: "*Space is constituted equally at all positions and in all directions; that is, equal constructions can be carried out at all positions and in all directions.* In its manner of expression this postulate immediately decomposes into two, one of which expresses the possibility of translation, the other of rotation…"[75]

The limitation to three dimensions expresses, according to Graßmann, the limitation of space. By defining physical space as homogeneous, isotropic and three dimensional, a view Graßmann links closely to the concept of movement, he anticipates ideas which – with no connection to Graßmann – appeared later in history in Helmholtz's conception of a geometry of motion.[76]

Helmholtz on physical space

"Thus it appeared that space, considered as a region of measurable quantities, does not at all correspond with the most general conception of an aggregate of three dimensions, but involves also special conditions, depending on the perfectly free mobility of solid bodies without change of form to all parts of it and with all possible changes of direction ; and, further, on the special value of the measure of curvature which for our actual space equals, or at least is not distinguishable from, zero. This latter definition is given in the axioms of straight lines and parallels."[77]

But Graßmann is incorrect when he assumes that these two principles sufficiently define the characteristics of real space. His misconception evidently is linked to his lack of knowledge concerning the relevance of the curvature in the make-up of space. Only if the curvature is zero, the Euclidean notion of space follows from the two principles Graßmann postulates.

In this case Graßmann failed – like many of the most exceptional geometers of his time – to solve the "main difficulty in these inquiries", namely the "readiness with which results of everyday experience become mixed up as apparent necessities of thought with the logical processes"[78]. At the same time Graßmann herewith denied himself every alternative access to non-Euclidean geometry. Only Riemann and Helmholtz managed, eleven and twenty-five years later and starting from analogous conceptions, to unveil this deeper form of understanding.

After his excursus into the foundations of geometry, Graßmann turns to mechanics. He effectively demonstrates the validity of vector addition for the treatment of movement of a system of point masses – determination of the center of gravity and of the resulting velocity, etc. The alternation which appeared in Graßmann's thoughts so far, between abstract inquiries into the *n*-dimensional affine point space on the one hand, and introductory and descriptive considerations on geometry and mechanics on the other, is a typical feature of the methodological structure of the *Extension Theory* of 1844.

Often Graßmann begins abstract reflections on *n*-dimensional space by constructing analogies in the framework of geometrical perception, and these take him to concrete examples from the fields of geometry and mechanics.

He also follows this path when he introduces the exterior multiplication of vectors. The § 28 of *Extension Theory* begins with the following words: "We begin with geometry in order to obtain from it an analogy for the way the abstract science must proceed, and at the same time so as to have in mind a clear idea of the unfamiliar and often difficult course along which the abstract science leads us."[79]

In impressive conceptual developments and always following the outlines of his general theory of forms Graßmann gradually obtains the exterior product of *m* vectors in the *n*-dimensional vector-space by defining oriented areas of parallelograms as the non-commutative product of the vectors spanning them. He then interprets the new mathematical objects (multivectors)[31] produced by this method of conjunction as oriented parallelotopes. Graßmann also demonstrates the distributivity of exterior multiplication with respect to vector addition and the change in sign with the swapping of arbitrary factors.

In his philosophical and methodological preliminaries, Graßmann emphasized that he considered *Extension Theory* to be the continuous counterpart of combination theory.[80] This parallel structure of *Extension Theory* and combination theory is expressed mathematically in the close connection between the exterior products of vectors and the theory of determinants on the one hand, and the implicit group-theoretical dimension in the entire *Extension Theory*, on the other.[81] To a certain extent, Graßmann became aware of this fact in the *Extension Theory* of 1844 when he illustrated the application of exterior multiplication by solving systems of linear non-homogeneous equations[82] and thereby obtained the same formulas as the theory of determinants.

17 years later, in his completely revised and augmented edition of *Extension Theory*, he explicitly showed how the theorems of determinant theory could easily be proven thanks to the exterior product of *n* vectors of the *n*-dimensional vector-space. It was Graßmann's approach to determinant theory that aroused Hankel's special interest and prompted an intense correspondence between the two scientists.[83]

In the further elaboration of his exterior vector algebra, Graßmann takes the logical step of analyzing the addition and exterior multiplication of multivectors ("addition and multiplication of magnitudes of higher order"). Considerations on the possibility and specificity of an inverse operation to exterior multiplication then follow this step, a procedure Graßmann calls "exterior division".[84] By showing the uniqueness of a "quotient of equal magnitudes", that is, the quotient of two multivectors spanning the same subspace, *Extension Theory* leads him to introducing number magnitudes as "magnitudes of order zero". This way of proceeding illustrates Graßmann's intention of constructing his theory independently from other mathematical disciplines, especially arithmetic.

When it came to solving the objective problem which lay in reconstructing geometry, Graßmann, like Möbius, in some respects was far ahead of his contemporaries. His way of handling geometry completely without recourse to metrics as an affine geometry – metrical approaches would only appear in the 1847 *Geometrical Analysis* in the context of introducing the inner product of vectors (scalar product) – and also his construction of geometry without relying on number theory underline this fact.

Graßmann's critique of geometry: on the relation of measurement, number and magnitude

"An essential defect of previous presentations of geometry is that one usually returns to discrete numerical ratios in the treatment of similarity theory. This procedure, which at first seems simple, soon enough becomes entangled in complicated investigations concerning incommensurable magnitudes, as we have already hinted above; and the initial impression of simplicity is revenged upon problems of a purely geometrical procedure by the appearance of a set of difficult investigations of a completely heterogeneous type, which shed no light on the essence of spatial magnitudes. To be sure, one cannot eliminate the problem of measuring spatial magnitudes and expressing the results of these measurements numerically. But this problem cannot originate in geometry itself, but only arises when one, equipped on the one hand with the concept of number and on the other with spatial perceptions, applies them to that problem, and thus in a mixed branch that one can in a general sense call by the name "theory of measurement"... To relegate the theory of similarity, and even that of surface area, to this branch as has previously occurred (not to the form but to the substance) is to steal the essential content from what is called (pure) geometry."[85]

For Graßmann there were two approaches in geometry, going back to Euclid, that served to avoid the concept of number. On the one hand, these were the intercept theorems; on the other, theorems on the equal areas of triangles (parallelograms) which have an angle in common and in which the including sides of this angle are inversely proportional. If either approach "is based on a *definition* of the proportion between displacements, then the fundamental properties of the proportions are expressed in the *statements on parallelism or about the surface-area*, and so these must be proven directly without recourse to the concept of ratio and, therewith, the concept of number."[86] Graßmann simultaneously found a solution to both problems thanks to the distributivity of exterior multiplication. Hereby, fully aware of the importance of his critique of geometry, Graßmann made a contribution to the axiomatic foundation of geometry, which was brought to a relative conclusion only in 1899 by Hilbert (Hilbert 1900).

On the last pages of the first section of *Extension Theory* Graßmann introduces the concept of projection ("shadows") of vectors and multivectors on a given subspace. He also shows that all vector-conjunctions introduced so far possess invariance with respect to this new operation.[87] Building up on this, he deals with the transformation of coordinates in *n*-dimensional vector-space and creates the equations for affine transformations in three-dimensional point space.

While in the first section of *Extension Theory* all investigations were based on the concept of the displacement, Graßmann constructs affine geometry in the second section on the fundamental notion of the point.

His initial basis is the addition of a given number of fixed position vectors ("deviation of elements from a given element"). Now since the final point of the system's position vector ("total deviation") is independent from the chosen starting point of the position vector, he directly posits the addition of the points, instead of the vectors. This is how Graßmann, starting from the concept of the vector, obtains the same "addition of points" that Möbius had developed in his *Barycentric Calculus* in 1827 (Möbius 1827).

Möbius' work had a clear influence on the general make-up of the *Extension Theory's* second section.

Just like Möbius, Graßmann assigns weight to the points in space, which he then interprets as the coefficients of these points, thereby making them applicable to the concept of magnitude. The geometric sum of such a system of points characterized by weight will then be the center of gravity of the system. Graßmann takes simple points as having the weight 1. Generally speaking, he therefore also considers the sum of the coefficients in a point-system to be 1 and, as a consequence, stays completely within the framework of affine geometry.

Graßmann creates the term "elementary magnitude" as the basic concept for all ensuing investigations. He defines this concept as follows: "Each structure is fixed as a magnitude once the range of its equality and difference is specified. Thus we regard two elementary assemblies [that is, systems of weighted points – H.-J. P.] as equal magnitudes, and indeed as equal *elementary magnitudes*, if their deviations from the same element always have equal values. An elementary assembly is therefore an elementary magnitude to the extent that one disregards the particular nature of its composition and retains only the value of the deviation that it forms with other elements, so that an elementary magnitude can be presented in another way as an elementary assembly, and each elementary assembly is interpreted as a particular embodiment of an elementary magnitude..."[88]

In his further developments, Graßmann searches for elementary representatives of elementary magnitudes. If the sum of the weights of a point-system does not equal zero, then the system may be replaced by its center of gravity, and for Graßmann the latter therefore turns into a simple representative of an elementary magnitude. If the sum of the weights equals zero, in turn, the sum-point represents an "infinitely" removed point of zero mass, which always may be represented as the well-defined difference $B - A$ of two points of equal weight. Graßmann now interprets this difference as an oriented, but spatially freely translatable vector (displacement, "Strecke") from point A to point B and he designates it as the representative of an elementary magnitude with weight "zero". Herewith, the description of elementary magnitudes of the first order is complete: they take on the form of weighted points and free vectors. This way of developing his concepts made it possible for Graßmann to embed his investigations of vector calculus in the *Extension Theory's* first section into his theory of elementary magnitudes as a special case.

The continuation of the second section therefore lies in the extension of the concept of a vector, which Graßmann accomplishes by introducing the elementary magnitude. He goes from treating oriented displacements in affine space as vectors to attributing vectors to points in affine space. These two kinds of vectors, or, in Graßmann's terms, "species of elementary magnitudes", can best be described by $(n + 1)$-tuples of real numbers. The vectors assigned to the points are represented here as $(n + 1)$-tuples in which the first position corresponds to the point's weight, whereas free vectors are represented by $(n + 1)$-tuples whose first position is always occupied by a zero. The process of adding these $(n + 1)$-tuples up becomes evident here. Graßmann relies on the first section's structure in his further developments.

Expanding the concept of exterior multiplication to include elementary magnitudes does not prompt new insights when all factors are free vectors. As Graßmann shows, all other cases in which factors either are only point-magnitudes or a composite of point-magnitudes and free vectors can be expressed as being the exterior product of a point-magnitude and a multivector. Graßmann designates the new magnitudes thus generated as "rigid elementary magnitudes". These magnitudes are simply bound multivectors – that is, the exterior product of two points or of a point and a vector gives a line vector ("line magnitude", "Liniengrösse"); the exterior product of three points or of two

Graßmann's definition of a "rigid elementary magnitude"

"Now we call a product of n elementary magnitudes of first order, or a sum of such products, an *elementary magnitude of n-th order*, and such a product whose elementary factors are not all displacements a *rigid elementary magnitude*. Thus we have proven the theorem that 'a rigid elementary magnitude of n-th order can be represented as a product of an element with an extension of $(n - 1)$-th order, and this extension, which we call the *divergence* of that elementary magnitude, is completely defined by it, but as its element any one belonging to the system defined by the elementary factors of the elementary magnitude may be adopted.'"[89]

points and a vector or of one point and two vectors gives a bound vector, i. e. a bivector that may be freely moved only on a plane ("plane magnitude", "Plangrösse"), etc.

In the context of discussing the exterior product of m points in affine space, there remains the question of defining the extension of the spatial magnitude representing this product. Graßmann defines this extension as the simplex spanned by the m points ("structure of vertices", "Eckengebilde"). By showing that the $(m - 1)$-dimensional parallelotope created by the exterior product of m points may be decomposed into $m!$ such simplexes, Graßmann reestablishes conceptual unity. In 1861 Graßmann abandoned this definition of the extension of a "rigid elementary magnitude", replacing it by the volume of a parallelotope because otherwise he would have encountered major difficulties when introducing the scalar product. In a remark in his 1861 second version of *Extension Theory* (A2), he wrote in this context: "One could also have defined the content of the surface element [*ABC*] as the area of the triangle *ABC*. But it will be shown below ... that then the content of the inner square of a displacement would only be half of the content of the square {of the length} of this displacement, whereas the two are in agreement with our nomenclature."[90]

After this topological digression[91], Graßmann gains remarkable insights into the different possibilities of constructing coordinate systems by applying his conceptual

Graßmann on coordinate definition

"If we take four rigid elementary magnitudes (that is, multiple elements) as fundamental measures, then we have the type of coordinate definition on which Möbius bases his *Barycentrisches Kalkül*... As reference systems of second order there appear six straight lines, each of which connects two of the reference elements, and which form the edges of a pyramid (tetrahedron) that has those reference elements as vertices; as reference domains of third order are four planes, each of which lie on three of the reference elements and form the faces of that pyramid; and the reference measures of second and third orders represent segments of those lines and planes. ... Every elementary magnitude of first order ... can be represented in space as a multiple sum of the four fundamental measures; every elementary magnitude of second order, whether it is a linear magnitude, a plane area of fixed direction, or an additive magnitude, can be represented as a sum of six linear magnitudes belonging to the six lines mentioned above; in brief, every magnitude can be represented as a multiple sum of reference measures of the same order, or as a sum of terms belonging to the reference domain of the same order."[92]

That means, when $X = (x_1, x_2, x_3, x_4)$ and $Y = (y_1, y_2, y_3, y_4)$ are the homogeneous coordinates of two points in projective space, then their exterior product is the sum of the six subdeterminants $x_1y_2 - y_1x_2, x_1y_3 - y_1x_3, x_1y_4 - y_1x_4, x_2y_3 - y_2x_3, x_2y_4 - y_2x_4, x_3y_4 - y_3x_4$, which are identical to Plücker's line coordinates *pjk*.

These important insights, which Graßmann obviously owed to Möbius' influence, also went unnoticed by his contemporaries. Alfred Clebsch first called attention to them when he wrote the obituary for Julius Plücker (1871).[93]

creations to geometry. We should emphasize here that Graßmann, with his principles, could easily generate homogeneous (and non-homogeneous) point, line, and plane coordinates, thereby particularly being ahead of Plücker in defining line and plane coordinates. Nevertheless, this is not to say that Graßmann was the first to pursue line and plane geometry: Chasles, Plücker, and others preceded him.[94] But he already possessed an exact conceptual definition of line and plane coordinates[95], thereby being two years ahead of Plücker. In his *Extension Theory* of 1844 Graßmann even presented the condition equation for line coordinates, which is usually attributed to A. Cayley (1860)[96], giving it in passing as an intermediate result in an investigation of mechanics.[97]

After basically applying the conceptual constructions from the first section of *Extension Theory* in a more general way to elementary magnitudes in the second section, Graßmann introduces, after dealing with the problem of coordinates, a new form of generating products with these vector magnitudes. In quite difficult constructions, characterized by an excessive striving for extreme abstraction,[98] he obtains a conjunctive product which is contrasted as a dual form of the exterior product and which he calls "eingewandtes Produkt" (a conjunction which later, in the 1861 revised edition of *Extension Theory*, is termed the "regressive product" and comprehensively developed there).

Graßmann connects his reflections to dimensional considerations on the generating system of two multivectors A and B entering into a multiplicative conjunction. Thus he obtains the fundamental relation: $\dim(A) + \dim(B) = \dim(A + B) + \dim(A \cap B)$, that is, the sum of the dimensions of the generating systems of A and B is equal to the sum of the dimension of the sum-space ("umfassendes Gebiet") and the dimension of the intersection space ("gemeinschaftliches Gebiet") of the generating systems of A and B.[99] If one now equates the dimension of the initial space ("Hauptgebiet") to n, then the exterior product of the multivectors A and B will only be unequal to zero if $\dim(A \cap B) = 0$ and $\dim(A) + \dim(B) = \dim(A + B) \leq n$, that is, when "$A$ is completely outside the domain of B" (hence the term "exterior product"). For the product AB we can then say:

$$\dim(AB) = \dim(A) + \dim(B) = \dim(A + B)$$

Now Graßmann requires explicitly that the regressive product also have a finite value, that is, a value different from zero, if the intersection space of the generating systems of A and B is not empty. In analyzing the conditions for such a product, he concludes that for a regressive product that does not equal zero the following conditions must hold: $\dim(A \cap B) > 0$, $\dim(A) + \dim(B) > n$ and $\dim(AB) = \dim(A) + \dim(B) - n = \dim(A \cap B)$. So while exterior multiplication generated a multivector belonging to the sum-space of the factor, the regressive product generates a multivector of the intersection space. We can clearly see the duality of the two conceptual constructions here.

Graßmann obtains the "full significance of the regressive product"[100] of two multivectors by transforming the given product in such a way that the first factor appears as a multivector of dimension n and the second factor as a multivector of dimension $(A \cap B)$. Then, the first factor is replaced by a real number representing the volume of the corresponding n-dimensional parallelotope. Herewith the regressive product of two multivectors has been completely defined and its relation to exterior multiplication has been established. The introduction of a norm for the multivector of n-th order ("Hauptmaß") already takes us beyond the context of the *Extension Theory* of 1844. In 1861, Graßmann used it to derive directly the scalar product of vectors and multivectors. By introducing the concept of the completion of a multivector, this later work presents more elegant and concise constructions concerning the regressive product.

By having established the concept of regressive multiplication, Graßmann is now capable of defining any given product of multivectors with m factors ("bezügliche Produkte von Elementargrössen"). Generally speaking, there are neither rules for the exchangeability of factors, nor for associativity, in these products. The product and eventual expressions in brackets are evaluated step by step, from left to right, so that either exterior or regressive multiplication, depending on the two magnitudes' order, can be applied to the multivector determined by the preceding factors' product and the following factor. So-called "pure products" represent a special case in products of multivectors with m factors. In these products, all factors will be subject only to either exterior or regressive multiplication. Pure products consisting of vectors or multivectors of the $(n - 1)$-th order have a special relevance here. They are characterized by complete reciprocity, that is, the entire exterior algebra of V^n can be constructed from the n linearly independent vectors or multivectors of the $(n - 1)$-th order thanks to the exterior and regressive products.

Applied to geometry, the exterior and regressive products of bound magnitudes correspond, respectively, to the constructions of the partition and connection of points, oriented displacements and oriented plane elements. This is how Graßmann, in the context of affine geometry, can account for algebraic descriptions of operations which correspond to the dualistic positional relations of projective geometry. So we obtain the following formulations for space: The exterior/*regressive* product of two points/*elements of a plane* determines exactly a part of the connecting line/*line of intersection* of the points/*elements of a plane*. The exterior/*regressive* product of three points/*elements of a plane* which are not situated on a line/*do not pass through the same line* determines exactly one plane element of the connecting plane/*the point of intersection of the line* *elements*. And finally, for planimetry, the following must be introduced: The exterior/*regressive* product of two points/*line segments* determines exactly a segment of the connecting line/*the point of intersection of the line segments*.

The consequence then is, especially for planimetry, that all constructions which are carried out only with a straightedge – that is, which are the effect of the partition and the connection of points and displacements – may be described by products with multiple factors of point and line vectors, in which the exterior and regressive multiplication of factors alternate (in Graßmann's terminology: "mixed products", "gemischte Produkte"). Graßmann puts it this way: "Indeed every construction using a straightedge in the plane consists either in connecting two points by a straight line or intersecting two straight lines; but the straight line between two points is their product, and the intersection point of two straight lines is likewise their product, if the weight is not taken into account. Consequently, for every straightedge construction in which a point or a line is used I can substitute a multiplication with this point or line; ..."[101]

Now in particular, if in the multiple-factor relative product of point and line vectors the sum of the orders of the factors is congruent to zero modulo n (n = number of independent points of the corresponding space, i.e. for a plane $n = 3$), its value will be a real number and may be represented, with the transition to the coordinates, by a single algebraic equation in the coordinates of the relevant geometrical objects. If, furthermore, this product is taken to equal zero, the real coefficients of the factors, that is, the point and line vectors, are also taken out of consideration and the product becomes the immediate algebraic representative of the straightedge-constructions, which in turn may be expressed in a coordinate equation.

Graßmann calls such products "planimetric products". Taking these considerations as a point of departure, Graßmann obtains a fundamental principle for generating all plane curves of any given order. For if in a planimetric product a variable "point" x appears n times as a factor, because the corresponding coordinate equation is an algebraic equation of the n-th degree in the coordinates of the point c, this product will define the point's geometrical locus as a curve of n-th order. This is how Graßmann obtains his main theorem on plane algebraic curves: "If the position of a point (p) is so constrained that three points, which result by constructions using a straightedge from that point (p) and a given series of fixed straight lines or points, lie on a single straight line (or if three such straight lines go through a single point), then the locus of that point (p) describes an algebraic curve whose order one finds by simple enumeration. Thus one has only to enumerate how often one returns to the moving point (p), without returning to another moving point; the number (m) so obtained is then the order of the curve."[102]

Graßmann would prove the converse of this theorem, namely that any algebraic curve may be generated by such a construction, in 1851.

In the *Extension Theory* of 1844 he demonstrated the first part of the main theorem only by referring to the example of the "geometric equation for curves of second order".[103] Here the planimetric product takes on the form $xaBcDex = 0$ (x… variable point, a, c, e… fixed points, B, D… fixed lines). If one introduces the values of the coordinates into the corresponding magnitudes, one obtains a general algebraic equation of the second degree in the coordinates of x. The geometrical interpretation goes as follows: If the straight line running through x and a is intersected with the line B, and if the straight line through the point of intersection and a fixed point c is intersected with the straight line D, and if, finally, the point x is on the connecting straight lines of the fixed point e with the point of intersection on D, then x is a point of the conic section, the parameter of which is determined by the fixed magnitudes a, B, c, D and e. In other words: x is a point of the conic section in question when the sequence of lines defined by the points a, c, e and the straight lines B, D comes together on itself. This mode of generation corresponds to the well-known Maclaurin construction of conic sections, which relies on generation by projective pencils. But while the generation of algebraic curves of higher orders than 2 by projective pencils of rays only leads to special cases, Graßmann's mode of generation remains universal. Graßmann himself gave the obvious contextualization of the planimetric product into projective geometry in 1851.[104] In order to do this, Graßmann introduced the concept of "higher projectivity", meaning that associated pencils of higher-order curves take the place of projective pencils of rays. He proved that the curve of $(n + m)$-th order generated by a planimetric product is identical with the intersection of two projective pencils of curves of n-th, and respectively, m-th order.

When it comes to formulating the theorem on the projective generation of higher order curves, which is generally attributed to M. Chasles (for $m = 1$, $n = 2$, 1853) and E. de Jonquières (arbitrary m, n, 1858) and named after them,[105] Graßmann should clearly be given priority.

Since curves of orders higher than two cannot be completely constructed by a straightedge any more, Graßmann develops the concept of the "lineal mechanism" from the geometrical interpretation of the planimetric product. According to Klein, this designates "…a system partly of fixed, partly of movable straight lines and points, with the movable straight lines necessarily passing through certain (not necessarily fixed) points, and the points necessarily moving on certain (not necessarily fixed) straight lines"[106]. So Graßmann's main theorem revolves around the statement: "A curve is algebraic when it can be generated by a lineal mechanism."[107] Lineal mechanisms – which shouldn't be confused with linked and *coulisse* mechanisms, investigated in detail by P. L. Čebyšev, prompting him to develop his theory of approximation[108] – are only of theoretical interest because they always "wobble" in practical applications. Graßmann's theory of curves was, as far as I know, not continued after 1900.[109] Though Graßmann had only given one example for the generation of second-order plane curves in the *Extension Theory* of 1844, he already presented the general theorem on the possibility of generating hypersurfaces of any order through the relative product, mentioned above, and proved it in the following form: "if we have an arbitrary equation between extensive magnitudes, whose terms are mixed products, the degree of the equation with respect to one of them (P) is always as great as the greatest number (m) of times this extensive magnitude (P) appears in a single term of nonzero magnitude, whence it would be replaced by numerical equations, at least one of which would attain a degree equal to that number, with respect to the indicators of the variable extensive magnitude."[110]

In the following years, Graßmann published no less than 12 articles[111] on this theorem's application in the generation of plane algebraic curves and algebraic surfaces. The generation of cubic curves and surfaces bears Graßmann's name today.

The *Extension Theory* of 1844 ends with a chapter concerning geometrical relationships, connected with considerations of Möbius' in his *Barycentric Calculus* (Möbius 1827), and a note on so-called "open products".

Luigi Cremona on Graßmann's theory of curves

Cremona made enthusiastic remarks on Graßmann as early as 1860. He called him an "éminent géomètre allemand" and commented appreciatively on Graßmann's constructions in (Graßmann 1846):

"A l'occasion de ces théorèmes qui se rapportent à la *géométrie des intersections*, je ne pui m'empêcher de mentionner une méthode très-expéditive et très-curieuse, dont la première idée paraît appartenir à Leibniz, mais qui a été vraiment établie par M. Graßmann dans un ouvrage intéressant [he is referring to the *Extension Theory* of 1844 – H.-J. P.] ... je ne sache pas que quelque géomètre ait donné aux recherches d. M. Graßmann l'attention qu'elles méritent."[112]

In his foreword to his *Barycentric Calculus* (Möbius 1827), Möbius had made his position clear, stating that the theory of relationships is a science "...which in the sense of the word chosen here encompasses the foundation of all of geometry, but which will also be one of the most difficult sciences, should it be presented in total generality and completeness."[113] So it seems only natural that Graßmann should appreciate a theory thus characterized by Möbius, when Graßmann was aiming to criticize the foundations of geometry and to reach the highest possible level of generality in his mathematical results.

His point of departure in treating geometrical relationships is the concept of affinity. He starts his developments with a notion of a space made up of points with mass, seeing points sharing the same spatial location, but unequal in mass, as different geometrical objects. He defines the concept of affine relationship for these "elementary magnitudes" in a vectorial and algebraic way (but without using the concept of transformation): Two assemblies of magnitudes are, according to Graßmann, affine to one another "if every numerical relation obtaining between magnitudes of one series, whichever it is, also prevails between the magnitudes of the other series..."[114]

If one increases the dimension of an affine point space by 1, that is, takes the spectrum of real "mass values" as an additional dimension, Graßmann's definition of affinity becomes the generally accepted definition. On the other hand, Graßmann's definition of affinity also becomes this common notion when all masses are normalized to 1. In this context, F. Klein also speaks of Graßmann's "naïve interpretation" of the homogeneous coordinates of $\mathbf{R}^n$ as affine coordinates of $\mathbf{R}^{n-1}$, while emphasizing this interpretation's extraordinary importance for applications in physics.[115]

Finally, Graßmann obtains the concepts of direct and reciprocal affinity by showing that only the exterior products of vectors and the regressive products of $(n-1)$-order vectors are invariant in relation to affinity.[116]

Graßmann derives projective relationship from affine relationship by examining subspaces: if two assemblies ("Vereine") of "weighted" points, line elements etc. are related affinely, then the corresponding affine subspaces ("Systeme"), whose elements ("Punkte, Linienteile") only differ in weight, will be related projectively ("linear verwandt"). This approach corresponds to the transition to homogeneous coordinates, according to the principle of projection and intersection.

Felix Klein on the connection between projective and affine relationships

In his discussion of the geometrical objects constituted by Graßmann, F. Klein writes: "These 'affine' and 'projective' interpretations of invariant theory naturally are not fundamentally opposed, but may be geometrically derived from one another according to the so-called principle of projection and intersection. The projective interpretation in $\mathbf{R}_3$ arises from the affine interpretation in $\mathbf{R}_4$ when one projects the zero starting figures from 0 (zero) onto any given $\mathbf{R}_3$ embedded in $\mathbf{R}_4$. There, the displacement starting from 0 gives a point in $\mathbf{R}_3$, the two-dimensional linear structure starting from 0 gives a straight line in $\mathbf{R}_3$, etc."[117]

According to Graßmann, direct affinity corresponds to collineation ("kollineare Verwandtschaft") in projective space; reciprocal affinity corresponds to correlation ("reciproke Verwandtschaft"). [118]

In passing, Graßmann touches on metric invariants in collinear transformations, which for him arise organically from exterior division. The cross ratio of four straight lines in space[119], so far unknown in projective geometry and bearing Graßmann's name today[120], is among the five cross ratios ("Doppelquotienten") he introduces.

This chapter's theoretical investigations conclude with some theorems on harmonic points and equations. By applying these results to crystallography, Graßmann explicitly shows for the first time the connection between the so-called zone law and the law of rational indices, by elegantly deriving the latter from the former.[121]

The closing paragraph of *Extension Theory* is on the concept of the "open product". Here, Graßmann treats a sum of mixed products of multivectors, which all have a multivector as a common factor. Given the fact that the rules of exchanges of factors in mixed products are very limited, the common factor usually cannot be factored out. But factoring out is possible in a formal sense when one factor is "pulled out" and the "gap", into which the factor must enter, is marked in the remaining products. Graßmann terms such formal products with gaps or linear combinations of such products "open products". He only demonstrates simple relations in dealing with open products by giving the example of an open product of vectors and bivectors in $\mathbf{R}^3$ with a gap. Graßmann saved the more general and comprehensive development of the concept, which he underlined as being extremely important for optics and mechanics[122], for a later treatise.[123]

The open products Graßmann develops, in modern terminology, are tensors. Graßmann's example of an open product represents a symmetric tensor of the second order. He already emphasized its independence from its basis representation.[124]

In the second, completely revised 1861 edition of *Extension Theory*, open products, now termed "gap products" or "gap expressions" ("Lückenprodukte", "Lückenausdrücke"), are treated thoroughly. This text gives us the derivation of the essential laws for operating with symmetric and skew-symmetric tensors of any order.[125]

With the remarks concerning open products, the circle of topics Graßmann treated in 1844 is complete. Thanks to his specifically constructed repertoire of instruments from linear algebra, Graßmann gained deep insights into the structure of projective and affine geometry of *n*-dimensional space.

Graßmann was planning to extend his methods to metrical geometry, that is, to develop the theorems linked to the inner product ("scalar product") of vectors and multivectors in a second volume to *Extension Theory*. But this volume, which was meant to unfold the unpublished ideas from his 1840 examination thesis, never appeared because of the general lack of appreciation for the *Extension Theory* of 1844.

But we can view his prize-winning contribution *Geometric Analysis, linked to the Geometric Characteristic invented by Leibniz* (PREIS), published in 1847, as a substitute.

3.4 The prize-winning treatise on geometric analysis (1847)

Graßmann was the perfect candidate for responding to the prize-question which had been set up to commemorate Leibniz' 200th birthday. The task was to reconstruct, as far as possible, and to further develop the fragments of the *Geometric Characteristic*, which Leibniz had left unfinished. Just as Graßmann had advertised his new geometrical method, Leibniz had marveled at his own discovery in almost the same words. In a letter to Huygens dated 8 September 1679, in which he offered some samples of his characteristic[126], we read: The *Geometric Characteristic's* main advantage compared to Descartes' analytical geometry "lies in the reasoning which can be done and the conclusions which can be drawn by operations with its characters, which could not be expressed in figures and still less in models without multiplying these too greatly, or without confusing them with too many points and lines in the course of the many futile attempts one is forced to make. This method, by contrast, will guide us surely and without effort. I believe that by this method one could treat mechanics almost like geometry."[127]

Graßmann on the reconstruction of the Leibnizian characteristic

Referring to Leibniz, Graßmann wrote in the introduction to his prize-winning treatise:

"In order to illuminate the scientific significance of his curious characteristic, and also in order to make us aware of this aspect of his scientific contribution, I will, in the derivation and development of the new analysis, take the Leibnizian characteristic as a point of departure and show how, from this seed, by a consistent process of development, by the proper deletion of extraneous materials and the fertilization with the ideas of geometric relationships, we obtain the analysis which I am inclined to regard, if only preliminarily, as the realization of the Leibnizian idea of a geometrical analysis. That this is not the path on which I arrived at this analysis, is hardly worth mentioning."[128]

In his prize-winning treatise Graßmann goes on to show how the Leibnizian approach, which he had so far been unaware of[129], when adequately modified, is included in the *Extension Theory's* geometric-algebraic constructions when these are extended to include the concept of the inner product.

We can give the following rough sketch of the contents of Graßmann's treatise: Graßmann takes the ideas of Leibniz as a point of departure. Leibniz, in turn, connects his inquiries on the geometrical characteristic to a peculiar symbolism serving to designate congruent systems of points, which may be carried over into one another by motion. That is to say that, if a, b, c are fixed points, x is a variable point and, as in Leibniz, "*8*" is taken to be the symbol for congruence, then the formula *a x 8 b c* implies that by a movement the displacement *ax* may always be brought into congruence with the displacement *bc*. So this formula defines the geometrical position of all points *x* as the surface of a sphere with the center *a* and the radius *bc*. In an analogous way, the formula *a x 8 b x* defines a plane as the geometric locus of all sphere-centers *x* passing through two fixed points *a* and *b*, and *a x 8 b x 8 c x* defines a straight line as the geometrical locus of all sphere-centers *x* passing through three fixed points. It can be shown that this procedure makes it possible to reduce the entire Euclidean geometry to a sphere.

We should note in this context that the idea of reducing geometry to a sphere played a significant role in the inquiries into the fundamentals of geometry in the 19th century. We find this procedure in Lobačevskij, Bolyai, Helmholtz, de Tilly and Lie. These mathematicians either did not, or could not, rely on Leibniz for this.[130]

Graßmann's criticism does not focus on the method used by Leibniz, but on his symbolism. He discards it because, according to Graßmann, in such expressions of congruence, "one can by no means always replace each expression with its congruent"[131]. In a second step Graßmann criticizes Leibniz for immediately relying on the concept of congruence, because, as *Extension Theory* had shown, entire areas of geometry could still be constructed without it (that is, projective and affine geometry, among others). Therefore Graßmann assumed a hierarchy of geometric relationships in his development.

By examining collineation between six points of space and the affinity between four points of space, and by analyzing the representation of their corresponding stereometric (planimetric) products, or exterior (regressive) products of point-, line- and plane-magnitudes, he returns to dealing with the relation of congruence from a higher standpoint.

Graßmann shows that Leibniz' simplest formula of congruence *a b 8 c d* amounts to requiring that the length of the displacements *ab* and *cd* be the same. So if congruence should be the case in the context of his vector calculus, so must the vector equation $a - b = c - d$ ($(a - b)$ and $(c - d)$ being the vectors from *b* to *a*, and from *d* to *c*). Consequently it must be possible to represent the length of the displacement *ab* or *cd* as a "geometric function"[132] of the vectors $(a - b)$ and $(c - d)$, and the Leibnizian formula *a b 8 c d* is replaced by the equation $f(a - b) = f(c - d)$. Graßmann sees this vector function and the elaboration of the theory connected to it as the appropriate reappearance and completion of the Leibnizian geometric characteristic.

Hereby Graßmann had already dealt with the dubious aspects of the Leibnizian characteristic. Firstly, he had replaced the geometrical relation of congruence by an equation, which made it possible to always substitute equal magnitudes in the equation. Secondly, this equation had not appeared abruptly as in the Leibnizian formula of congruence, but it was contextualized in his newly created theory of vector algebra.

The following quotation illustrates very well Graßmann's steps towards constructing the vector function he was looking for, which would reveal itself to be the inner vector product (or scalar product of vectors): "To find such a function", Graßmann wrote, "we first assume that all the magnitudes in question lie on the same line. Consequently two equally long line segments can only have the same or opposite directions. … If p denotes a displacement, then p is as long as $(-p)$; but besides p and $(-p)$ there is no displacement on the same straight line with length equal to that of p. It would thus follow, but only here, that $f(p)$ must have such a form that $f(p) = f(-p)$. If one could treat displacements on a line of numbers, then p^2 would be such a function, and in fact the simplest to satisfy this condition. Thus we must next investigate whether and to what extent the laws of numerical conjunction can be applied to displacements on the same line…"[133]

This latter investigation presented no problems to Graßmann since in 1844 he had already introduced numbers through the concept of the exterior quotient. If one takes the divisor of such an exterior quotient as the common measure, then all vector equations may be expressed. Thus Graßmann obtains the definition: "By *inner products of any two parallel displacements* I mean magnitudes which are regarded as proportional to the numbers that result if one measures the two parallel displacements of one of those inner products by the same unit and multiplies the quotients of these two measures, all units however assumed to be equally long. The inner product of two displacements is symbolized by $a \times b$, the inner square $a \times a$ by a^2."[134]

Here we will have to note that Graßmann *does not* consider "inner products of magnitudes of the same order" to be numerical magnitudes, but only to be "proportional to a series of numerical magnitudes"[135], which then are not further elaborated. Only in 1861, in the new version of *Extension Theory*, the inner product of multivectors of the same order is a real number.

Graßmann interprets the inner product of a vector from point a to point b as the square of the distance between points a and b. So by normalizing one vector from every one-dimensional subspace, respectively, and defining a corresponding scalar product, Graßmann introduces Euclidean metrics into affine point space.

Graßmann's inner product of parallel vectors turns out to be commutative and distributive in relation to vector addition. By requiring permanence for these characteristics and the validity of the Pythagorean theorem, he obtains the general form for the inner product of any two vectors: "By the *inner product $a \times b$ of two nonparallel displacements a* and b I will designate the inner product of the first, a, with the orthogonal projection of the second on the first."[136]

The concept of the "orthogonally proportional"[137] establishes the connection between the inner and the exterior product. Here the inner product of two vectors turns out to be proportional to one of the two vector's exterior product into the "orthogonally proportional surface area"[138] (that is, into the completing bivector). Graßmann replaced this still unfinished and somewhat impractical concept[139] in 1861 by the concept of the "complement", which he then elaborated strictly and completely.[140]

The connection with exterior multiplication made it possible for Graßmann to introduce the definition for inner products of any given multivector already in his prize-winning treatise.

The remaining sections of the treatise consist of applications in geometry and mechanics, as well as a presentation of some formal concepts (such as the "inner product of points"), which are excluded from later works for their lack of viability.

The completion of this treatise in 1847 basically brought Graßmann's conceptual architecture of linear algebra to a conclusion. After affine geometry, Euclidean geometry in *n*-dimensional space had now become accessible to algebraic approaches. In 1861 Graßmann left behind the verbal-conceptual approach and the semi-formal mode of presentation with his new version of *Extension Theory*, which to this day is worth reading and perfectly readable once one has become familiar with Graßmann's terminology.

3.5 The *Extension Theory* of 1862

By reworking *Extension Theory* in the 1860s, Graßmann undertook one last effort to win the scientific community's acclaim for his findings, since they had been completely ignored up to that time.[141]

Graßmann submitted his approach to a radical reconstruction. He now omitted philosophical reflections completely. He abandoned the separation between extension theory and analysis, made calculating with concrete and abstract numbers a prerequisite and increased the diversity of the contents discussed in the book. "Thus ... in the present work", Graßmann wrote in the foreword, "I have presupposed the other branches of mathematics, at least in their elementary development. In addition I have adopted exactly the opposite method in the form of presentation, as I have applied the most rigorous mathematical form we know, the Euclidean, to the present work, and have relegated to the Remarks everything that serves to illustrate or motivate the method chosen."[142] But it took until the end of century for the book to receive the recognition it deserved – even though it then was immediately considered a methodological model.[143]

I will only give a brief outline of this work here, partly because the book's concepts have already been discussed above, partly because they have passed, though in different forms, into the repertoire of modern mathematics. It is preferable to use, as far as possible, modern terminology in order to identify Graßmann's core reflections more clearly.

Starting with linear combinations over a system of linearly independent magnitudes ("Einheiten"), Graßmann develops, without referring to geometry, the fundamental concepts of vector calculus over $\mathbf{R}^n$. He defines the dimension of a vector space, analyzes basis transformations, defines the exchange theorem for basis vectors and determines the dimensional relations between the sum and the intersection of subspaces. He makes explicit the modular characteristics of vector addition and linearity related to combining vectors with elements of the field of real numbers. And he postulates, when investigating possible products between vectors, the additivity and homogeneity of the resulting combination as a principle of construction.

Taking the product of two vectors in basis representation as a point of departure, Graßmann emphasizes as uniquely significant two kinds of tensor products: the skew-symmetric and the symmetric kinds.

The first section of the 1862 *Extension Theory* analyzes skew-symmetric products and product-structures built up on them. In other words, it develops the exterior algebra over V^n.

In the second section, which treats the vector functions $X = F(Y)$ over V^n, the essential magnitude of the symmetric tensor appears as the "open expression with interchangeable gaps" ("Lückenausdruck mit vertauschbaren Lücken"). As in the exterior vector-product, skew-symmetric tensors come into play, that is, "open expressions with non-interchangeable gaps". A mixed open expression thereby becomes the general representative of a tensor.

The first section is an extensive development of the relations of the exterior algebra over V^n. By investigating multivectors, their relations and transformational invariances, Graßmann obtains, via applications in the theory of determinants and via solutions to linearly non-homogeneous systems of equations, the concept of the complement. As a logical consequence, Graßmann connects the concept of the inner and exterior scalar product of multivectors to the complement of a multivector. The scalar product of vectors, in turn, leads him to orthogonalize the n-dimensional vector space and to the concept of the orthonormal system ("einfaches vollständiges Normalsystem"). This gives him metrics and the definition of angle. Then Graßmann elaborates the invariance of vectors in relation to orthogonal transformations in the now "complete" *Extension Theory*. Graßmann's applications in geometry only come after this process of elaboration. They take place in analogy with the investigations into elementary magnitudes over $\mathbf{R}^3$, which we find in the *Extension Theory* of 1844. He treats planimetric and stereometric products and proves completely the main theorem on the generation of algebraic curves, mentioned above. Only the inner product of vectors is submitted to interpretation from a geometrical point of view, and thereby Euclidean geometry becomes part of a vector-algebraic treatment.

The second section discusses the properties of vector functions and their differentiation and integration. Graßmann uses the concept of the tensor, which takes on the form of an open product, as a methodological instrument.

Linear operators, which Graßmann calls "quotients", appear in the text as special "open expressions with a gap". Graßmann expands his analysis of their characteristic roots to complex operators and proves that it is possible to transform these operators into the triangular form.[144] He also emphasizes the importance of the quotients for characterizing affinity and collinearity.

Compared to the *Extension Theory* of 1844, geometrical relations remain in the background. But Graßmann presents a new form, namely an exemplary extension of Möbius' circle relationships. The investigation of the differentiation of vector functions with open products leads Graßmann to the concept of the Jacobian determinant.

Graßmann ends the book by developing the concept of skew-symmetric tensors and by applying it to systems of partial differential equations. Notably, Graßmann's application of the conjunction laws of skew-symmetric tensors historically represents the first complete grouping of all the possibilities arising from what we call Pfaffian differential equations. Thereby Graßmann indirectly highlighted the relevance of exterior algebra for differential geometry.

This enormous range of work, of whose ideas I have only given a rough sketch, is the masterpiece of Graßmann's mathematical creation. Graßmann had surpassed his contemporaries on many questions. In the process, he had not simply distanced himself from the 19th century's fundamental mathematical questions. Trained by the dialectics of Schleiermacher, Graßmann had been a deeper thinker and therefore he had offered solutions which were not equaled by "standard" mathematics, especially on the continent, until the 1870s and 80s.

Hamilton was one of the few people to understand the value of Graßmann's research at an early stage. Hamilton himself was a groundbreaking figure in the context of the English algebraic school of thought, and he possessed enough scientific authority to communicate his ideas to a wider public. We would also have to think of Hermann Hankel, who, in a mindset resembling Graßmann's, also received the acclaim he deserved only by the end of the century.

But before turning to the imprint Graßmann's ideas made on the living body of mathematics, we must take another of Graßmann's works into consideration. This work seems to have nothing to do with his previous investigations.

3.6 Work on the foundations of arithmetic (1861)

The work we are dealing with here is the *Textbook of Arithmetic for secondary schools* ("Lehrbuch der Arithmetik für höhere Lehranstalten"), the contents of which were the fruit of joint research by the brothers Hermann and Robert Graßmann. It was published by Hermann Graßmann in 1861. With its "strictly Euclidean structure" it predates the second *Extension Theory's* methodological shift by a year. It is also a display of the increasing influence of Robert's thoughts. But the book could not live up to what the title promised – its high level of scientific abstraction and mode of presentation made it unsuitable as a *teaching* manual for secondary institutions.

Instead, it was extraordinarily important from a scientific point of view; it contained nothing less than the most rigorous in-depth analysis and presentation of the foundations of arithmetic up to that date. On the one hand, these investigations were motivated by Graßmann's methodical and methodological views on the relations between the mathematical disciplines. On the other hand, observations concerning a philosophical foundation for mathematics also played a part. Both topics had already appeared on the first pages of the *Extension Theory* of 1844. This is to say that the necessity Graßmann had felt for finding a place for his *Extension Theory* in the structure of mathematics also led him to redetermining the location and structure of other mathematical disciplines.[145] Therefore the investigations on arithmetic became an organic part of the programmatic approach of 1844, and the seemingly arbitrary focus on arithmetic was a necessary consequence. It is worth noting that this approach had been breached and called into question by an intense occupation with the foundations of arithmetic. This fact has not received enough attention so far.

Felix Klein gives us a fitting description of Graßmann's point of departure: Graßmann opposed the view, Klein writes, "that geometry may become merely an application of arithmetic, and he claims for his Ausdehnungslehre the status of an independent science. And from it he distinguishes, again as an independent subject, 'mensuration' (Messkunde). The latter is built on arithmetic; and so it was altogether consistent for Grassmann now to occupy himself with the foundations of arithmetic. Thus he became one of the first to investigate the essential properties of ordinary calculation."[146]

If we now wanted to look for, in these conditions, the reason why Graßmann committed himself to an independent foundation of arithmetic, we would have to say that his philosophical views on the essence of mathematics – imparted by his father and modified by the effect of Schleiermacher's dialectics – were the decisive factor influencing his approach to the foundation of arithmetic. The following table briefly gives the significant points:

The philosophical foundation of the "constructive" structure of arithmetic	
Justus Graßmann: 1. The essence of mathematics consists in generating "its primary concepts by its inherent force of synthesis (which we will, in the wider sense, call a construction)" … "But it is not the form of this synthesis, but the product of it that is its subject."[147]	**Hermann Graßmann:** 1. "Thinking exists only in reference to an existent that confronts it and is portrayed by thinking; but in the real sciences this existent is independent, existing for itself outside thought, whereas in the formal it is established by the thinking itself, when a second thought-process is confronted as an existent."[150]

2. The theorems of mathematics "express the nature of this specifically mathematical synthesis, and of what is given along with it".[148]	2. "Pure mathematics is therefore the science of the *particular* existent that *has come to be* by thought. The particular existent, viewed in this sense, we call … a *form*; thus pure mathematics is the *theory of forms*."[151]
3. "The synthesis of the same kind of things gives us magnitude; it is *discrete when, in the process of its generation, what is to be combined* (through the synthesis of which the magnitude arises) is taken *to be given*;…"[149]	3. "Each particular existent brought to be by thought … can come about in one of two ways, either through a simple act of *generation* or through a twofold act of *placement and conjunction*. …that arising in the second way is the *discrete* or *conjunctive form*."[152]

This comparison shows clearly that Graßmann was never interested in an axiomatic presentation of arithmetic. Accordingly, he noted in the *Extension Theory* of 1844: "Although postulates have been introduced into the formal sciences, for example in arithmetic, this is to be regarded as an error, only to be explained by the corresponding treatment of geometry. … Here it is enough to have demonstrated that postulates are necessarily absent from the formal sciences."[153]

To Graßmann, mathematical theorems were statements about the content of constructions belonging to the active realm of thought. They were theorems concerning all ideal constructions which displayed the same structure in relation to well-defined criteria. In this sense, Graßmann must be seen as one of the earliest architects of constructive or operative mathematics.

The context, as discussed above, surrounding the *Textbook of Arithmetic* is not reflected in the work itself, as opposed to the 1844 *Extension Theory*. The textbook expresses the "conviction that it represents the first strictly scientific discussion of this discipline, with the even more far-reaching requirement that its method, however far it may stray from the common paths, should not be seen as one among many, but as the only possible option for a strictly coherent and adequate discussion of this discipline."[154] Nevertheless, no justification is given for this conviction because, according to the author, there is no room for discussing methods in a schoolbook: "We hope to make up for this shortcoming at some later date by presenting a discussion of mathematics which, presupposing scientifically educated readers, will explain all its guiding thoughts and show the necessity of its methods in every detail."[155]

The general tone alone in this foreword already differs significantly from what we are used to hearing from Hermann Graßmann. While his father Justus considered dialectical progression, the identification of a guiding idea and the feeling for an organic wholeness in a mathematical concept to be fundamental especially for school purposes, far more important than any calculation or written proof,[156] now the pedagogical emphasis is laid on the "most rigorous method possible"[157]. Even though the foreword emphasizes, in its further course, the teacher's obligation to bear Justus Graßmann's points in mind, this does not in any way shape the mathematical mode of presentation. This explains why Carl Gottfried Scheibert, who had developed textbooks on trigonometry with Justus Graßmann, rejected and despised such a lack of pedagogy in a schoolbook.[158]

But how does Hermann Graßmann develop arithmetic?

He does *not* begin by generating numbers, but by generating a series of magnitudes, infinite on both sides, by connecting positive and negative units.

He opens the first section of the textbook of arithmetic by giving the following plan of construction:

"*Explanation*. Use the following procedure to generate a series of magnitudes from the magnitude e: Put e as a term of the series, put $e + e$ ... as the following term of the series, and continue like this by deriving the following term from the corresponding last term of the series by adding $+ e$ to the latter. In the same fashion, put $e + -e$ ... as the term of the series immediately preceding e, and so go on to derive from the corresponding first term of the series the immediately preceding one by adding $+ -e$ to the former. You will get a series which continues infinitely on both sides. ... If, in this series, one assumes every term to be different from all other terms of the series, then we will call this series the *basic series*, e the *positive* unit, $-e$ the *negative* unit." [underlined by H.-J. P.][159]

Recursively he defines addition for the thus-generated terms of the basic series, but still there is no mention of numbers. Only under §4, multiplication, theorem 52, he introduces the number 1:

"$a \times 1 = a$ ..., 'Multiplying by one changes nothing.'"[160]

In definition 53 we learn that a basic series whose unit equals 1 is called a number series. And definition 60 shows that the product of a number and a magnitude gives another magnitude of the same basic series. Finally, we learn in definition 64:

"If a is a magnitude from the basic series generated from $e \neq 1$, and $a = e\,\alpha$, then we will call a a named magnitude, e its unit, α its numerical value."[161]

Now, on page 21 of the textbook, Graßmann's basic idea becomes clear: arithmetic is no longer a "pure" science of numbers, as his father had still seen it, and he also does not bother with wondering whether 0 and 1 are numbers, whether addition may be viewed as an arithmetical operation, or whether the negative may be part of the numerical series; Graßmann's aim is to construct the "named number" ("benannte Zahl") as *the* object of arithmetic.[162] The derivation of whole numbers and rational numbers is just a necessary sideline of inquiry in this case. He constructs "named numbers" as *extensive magnitudes*: as one-dimensional vector components (at first integer, in a later part of the textbook also rational[163]), which had already been the basic element of construction in his *Extension Theory* of 1862.[164] The most natural point of departure is to begin by constructing an infinite cyclic group. The *Extension Theory* of 1862 confirms that

Hermann Graßmann views "named numbers" as extensive magnitudes: "Only if the system consists of just the absolute unit (1) is the derived magnitude not an *extensive*, but a numerical magnitude.

In general I will reserve the expression *magnitude* for these two species of them only."[165]

And he makes the following remark:

"From elementary mathematics we assume the laws of calculation for numbers, and also for the so-called "named numbers" {benannte Zahlen}, that is, for the extensive magnitudes derived from a single unit; but only for the case that that unit is an original unit."[166]

A profound change in the foundation of mathematics had taken place. The classificatory schema of mathematics, which had been developed in the *Extension Theory* of 1844, had collapsed completely: Now, extensive magnitudes were the sole object of mathematics, magnitudes which, in a special case – a basic series containing the unit "1" –, act as numerical magnitudes. The theory of numbers turned into a special case of arithmetic, which turned into a special case of extension theory – nonetheless, the most important special case. The general theory of forms, elaborated in the *Extension Theory* of 1844, had been surpassed. The classificatory schema of mathematics had to be redrawn completely (it would serve later, in Robert Graßmann's mathematical and logical publications, as an instrument to separate mathematics from logic). Mathematics was now detached from philosophy and moved even closer to the "objective sciences", the so-called "Realwissenschaften". It became the formal theory (and model theory) of all measurable entities, as we have already seen above in the considerations of Klein and Helmholtz (1887).

Robert Graßmann's recollection of this process shows that the Graßmann brothers were aware of these consequences. He wrote in retrospect: "In 1847, in a collective effort, the brothers Hermann and Robert Graßmann made a serious attempt to elaborate the theory of magnitudes. At the time, they chose single branches as their point of departure, namely the theory of forms or mathematics, the theory of numbers and extension theory, and the theory of combinations, which at the time they still took to be one of the branches of mathematics, and tried to generalize the operations belonging to these branches. But they did not manage to elaborate satisfactory theorems and therefore gave up their attempt at representing this branch."[167]

The fact that the Graßmanns' *Textbook of Arithmetic* played a fundamental role concerning the foundation of Graßmann's concept of mathematics has been overlooked, and this probably has to do with the book's history of reception and its great importance in the axiomatization of the theory of natural numbers: the text was viewed exclusively as a theory of numbers.[168] Only Victor Schlegel – in his 1878 Graßmann-biography – seems to have felt the text's general conceptual importance when he remarked in passing that the textbook served as a preliminary to the *Extension Theory* of 1862 "because it contains the elements of arithmetic which the latter presupposes..."[169]

Even if we limit ourselves to assessing the text's "casual" contribution to the foundations of arithmetic (unnamed numbers), the genius of this work's approach is impressive.[170]

With the conjunctive concept of the "addition of units"[171], Graßmann introduces, in examining the basic series, the concept of a's successor ("element immediately following a") as being $a + e$, and the concept of a's predecessor ("element immediately preceding a") as being $a - e$. He also defines zero as an abbreviation for $e + -e$. Graßmann's conceptual clarity suffers from the fact that from the beginning he uses the same symbol ("+") to define both addition and the successor. Here, he deviates from the formal symbolism for conjunction which he had developed in the general theory of forms in 1844.

Graßmann defines addition recursively, probably for the first time in history[172]: "When a and b are elements of a basic series, the sum $a + b$ must be the element for which the formula $a + (b + e) = (a + b) + e$ holds [second brackets added – H.-J. P.]."[173]

By referring to the basic series, this definition effectively allows Graßmann to construct all valid formulas of addition.[174] Therefore, addition is not an axiom for Graßmann. He inductively proves associativity and commutativity of addition, without thinking about the inductive proof's validity at this point.[175] After dealing with subtraction and the resulting theorems, Graßmann proves the permanence of all assertions made so far and gives definitions for calculi which have been derived from the basic series in the following way: "If, from a magnitude E which is not zero and which is part of the basic series, one derives a series of magnitudes in the same way as the basic series has been derived from e, then in the series thus generated every element will also be different from all others ... and all theorems constructed so far will be valid for this new unit and this new basic series."[176]

By introducing multiplication, Graßmann passes from the basic series to the number series. He writes: "We consider $a \,.\, 1$ (reads a times one or a multiplied by one) to be the magnitude a itself, that is, $a \,.\, 1 = a$ ... A basic series whose unit equals one is a number series, the elements of which are numbers ..."[177]

Multiplication is generally defined recursively in the number series thus postulated: "Multiplication with the remaining numbers (excluding 1) is determined by the following formulas: ...

$a \,.\, (\beta + 1) = a\beta + a$ ...

$a \,.\, 0 = 0$ ...

$a \,.\, (-\beta) = -(a\beta)$ ..."[178]

Later, he proves inductively that the product of two numbers gives a number from the basic series.[179] On the following pages Graßmann gives proofs for associativity and commutativity of multiplication, distributivity with respect to addition, as well as introducing the "smaller than" and "greater than" relations.

The fact that Graßmann's definitions and theorems are sufficient for an axiomatization of whole numbers shows once more how profound his investigation was. Hao Wang extracted the following axiomatic system from the developments in Graßmann's textbook, also proving its identity with a comparable system from modern abstract algebra:

"Graßmann's calculus L_2:

A. Atoms: =, (,); *a*, *b*, *c*, *d*, etc. (letters); 1, +, −, .; Pos.

B. Terms: 1 is a term; −1 is a term; a letter is a term; if *s* and *t* are terms, then $(s + t)$ and $(s \,.\, t)$ are terms.

C. Definitions …:
(2.20) $0 = 1 + -1$.
(2.21) For any *a* and *b*, $a - b$ is the number such that $b + (a - b) = a$.
(2.22) $-a = 0 - a$.
(2.23) $a > b \leftrightarrow a - b \in$ Pos.

D. Axioms
(2.26) $a = (a + 1) + -1$.
(2.27) $a = (a + -1) + 1$.
(2.28) $a + (b + 1) = (a + b) + 1$.
(2.29) $a \,.\, 0 = 0$.
(2.30) $1 \in$ Pos.
(2.31) $a \in$ Pos. $\rightarrow a + 1 \in$ Pos.
(2.32) $b = 0$ or $b \in$ Pos. $\rightarrow a \,.\, (b + 1) = (a \,.\, b) + a$.
(2.33) $b \in$ Pos. $\rightarrow a \,.\, (-b) = -(a \,.\, b)$.
(2.34) If $1 \in A$, for all *b*, $b \in A \rightarrow b + 1 \in A$, and $b \in A \rightarrow b + -1 \in A$; then for all *a*, $a \in A$.
(2.35) If $1 \in A$, and for all *b*, $b \in A \rightarrow b + 1 \in A$, then for all *a*, $a \in$ Pos. $\rightarrow a \in A$."[180]

Hao Wang's reaction to Graßmann's *Textbook of Arithmetic*, explained above, requires us to make two remarks. When Hao Wang says about Graßmann's work: "This was probably the first serious and rather successful attempt to put numbers on a more or less axiomatic basis."[181], this is an overstatement of the axiomatic point of view which fails to take Graßmann's basic constructive outlook, mentioned above, into account.

And when Hao Wang also suggests that Graßmann failed to explicitly exclude the possibility that all whole numbers might be posited as identical among themselves, in order to comply with his implicit axiomatic structure, we must disagree with this objection because it ignores Graßmann's constructive point of departure.[182] We have already seen that Graßmann's definition of the basic series directly requires that all numbers be different: "If… every element is assumed to be different from all other elements of the series, then this series will be called the *basic series*…"[183], and all of Graßmann's following theorems and definitions explicitly refer to basic series. Thanks to the scientific efforts he had undertaken together with his brother, Graßmann was fully aware of this necessary precondition in the foundation of an arithmetic of natural (and whole) numbers.

As Robert Graßmann recalled later, the Graßmann brothers had already been working on a new foundation for mathematics in 1847/48 and 1855/56, partly in a joint effort, partly in a division of labor in which Robert Graßmann increasingly concentrated on logic. The conjunction of "single-valued magnitudes" formed the common ground for these inquiries. Both had gradually come to the conclusion that for mathematics only conjunctions having the form $e + e \neq e$ would be valid, and only conjunctions having the form $e + e = e$ would be valid for logic (amounting to logical conjunction or the Boolean algebra). So Hermann Graßmann must have been completely clear about the distinction of these two modes of conjunction. [184]

3.7 The impact of Graßmann's ideas on the development of mathematics

Hermann Hankel, in his 1867 work on complex number systems (Hankel 1867), was the first mathematician to refer to Graßmann's algebraic, geometrical and arithmetical investigations and to publicly recognize their importance.

Hankel, who was keenly interested in the history and the philosophical aspects of mathematics, had been introduced to modern mathematics by Drobisch, Möbius, Riemann, Weierstraß and Kronecker[185]. He also possessed a profound knowledge of the British mathematicians' algebraic and geometrical inquiries and was uniquely destined in his time to delve into Graßmann's mathematical thought. His 1867 contribution to axiomatizing algebra[186] is strongly characterized by the goal of finding a synthesis and general interpretation of Graßmann's and Hamilton's algebraic studies. It is not just that Hankel relied on Graßmann's geometrical interpretation of exterior algebra, highlighting its importance for an elegant approach to the theory of determinants, and that he took up Graßmann's fundamental insights concerning the construction of arithmetic. Also, the highly important first and second section of his theory on complex number systems, that is, the "Exposition" and the "General Theory of Forms"[187], have been strongly influenced by both of Graßmann's *Extension Theories*, especially the 1844 theory of forms. It is in these sections that Hankel develops the principle of permanence, proposes a more abstract understanding of mathematical objects and does essential preparatory work for his abstract approach to concepts such as group, body and extensive body.

Emphasizing Hankel's role in the development of abstract group theory, Hans Wußing wrote: "It should be emphasized that one of Hankel's aims in his *Theorie der complexen Zahlensysteme* … was to transplant Hamilton's calculus of quaternions to Germany in a 'transparent' … manner, that is, in the formalistic manner. This also provided the occasion for the adoption of the technical terms 'distributive,' 'commutative,' and 'associative' used in England since 1840. In spite of the fact that it does not include the word 'group,' the second section, 'Allgemeine Formenlehre,' is especially important for its relation to group theory. The same is true of § 4, 'Algorithmus associativer Rechnungsoperationen ohne Commutation'."[188] N. Bourbaki arrived at a similar conclusion.[189]

Nevertheless, there is hardly any word in the scientific literature on the great significance of Graßmann's ideas for Hankel's work.

But Hankel also suffered Graßmann's fate. Continental mathematics had not evolved far enough to be able to incorporate his abstract conceptions. Only in the 1870s and 1880s interest in and respect for Graßmann's and Hankel's work would arise, as well as for the groundbreaking ideas of Galois, Möbius, Chasles, Cayley, Plücker, Riemann, Lobačevskij, Boole, Bolzano and others, who had begun to uncover the deeper structures of mathematics with their abstract approaches.[190] For geometry this development process has already been indicated.[191] Generally speaking, interest in the foundational questions of mathematics increased tremendously in this period of time.

Cantor found his first foundational theorems of set theory, mathematical logic began with G. Frege and E. Schröder in Germany, B. Peirce in the USA and G. Peano in Italy, and considerable progress was made in the formalization of abstract algebra, etc.

Under these circumstances, Graßmann's achievements were recognized in retrospect and his ideas were taken up and reformulated.

Hankel's *Theory of Complex Number Systems* (1867)

The following quotations illustrate Graßmann's influence on Hankel's conceptual developments. They are self-explanatory and just a selection from a much greater number of examples:

On the general theory of forms:

"According to the principles we have explained here, pure formal mathematics is not a generalized case of common arithmetic; it is a completely new science, the laws of which may not be *proven* by arithmetic, but only *exemplified*…"[192]

"Such formal laws, which may be completely different from those of common arithmetic, may be submitted to a special propaedeutical investigation which will have to distance itself by abstraction from the current meaning of the operation in question. And this will reveal itself to be especially useful when the same laws recur with differing contents in different disciplines. This formal kind of mathematics, under the name of 'calculus of operations' or 'symbols', would then be identical with a discipline recently practiced with great enthusiasm by the English."[193]

Graßmann's merit:

"The idea of letting a pure theory of forms precede the theory of magnitudes, and of conceptualizing the latter from the standpoint of the former, as important as it may have been for the foundation and the structural architecture of mathematics, was for a long time essentially worthless for its further construction because we had not ventured beyond using it to prove theorems with which we have not just been long-since familiar, but which have also been abundantly discussed and justified, if only 'empirically'. Only H. Graßmann grasped this thought in a truly philosophical spirit and investigated it from a comprehensive point of view."[194]

1. Arithmetic

E. Schröder, in his *Textbook of Arithmetic and Algebra* ("Lehrbuch der Arithmetik und Algebra", Schröder 1873), relied on Graßmann's investigations into the foundations of arithmetic from 1861, using them in 1873 to build up the arithmetic of natural numbers. In 1887, Helmholtz, in his philosophically important essay *Counting and Measuring from an Epistemological Point of View* (Helmholtz 1887), had also begun to rely on Graßmann's approach. Graßmann's work, apart from Dedekind's findings, had been the immediate inspiration for Peano's arithmetic postulates (Peano 1889b).[195]

Even in 1914, O. Hölder, in his article entitled *A Rigorous Foundation of Arithmetic* (Hölder 1914), still relied heavily on Graßmann's way of proceeding. This goes to show how strong and sustainable Graßmann's investigations into the foundations of arithmetic were for mathematics.

2. Geometry

In this context, it is especially worth noting that Graßmann's inquiries into the invariances of affine and Euclidean transformations, which he undertook in both of his *Extension Theories*, still exerted an influence on F. Klein and the elaboration of his important Erlangen Program. Hankel and Stern had drawn Klein's attention to Graßmann's *Extension Theory*, and in 1871 he had undertaken a close study of Graßmann's work. He wrote about this experience in a letter from 1909 to F. Engel: "As we know, Graßmann in his *Extension Theory* is an affine, not a projective, geometer. In the late fall of 1871 I had become aware of this and it led me (apart from my studies of Möbius and Hamilton, and apart from what I had gathered from my experiences in Paris) to conceiving what later would be my Erlangen Program. I distinguished (depending on the adjointed group) different 'methods' in geometry. ... In fall of 1872, while I was writing my Erlangen Program and while Lie was visiting me here in Erlangen, we quickly reached a level of mutual understanding. The word 'method' bothered him, and therefore I expunged it."[196] In the Erlangen Program we find multiple references to Graßmann concerning his circle transformations[197], concepts relating to *n*-dimensional manifolds[198] and to his indirect group-theoretical approach in developing *Extension Theory*[199].

A closer analysis of the Erlangen Program also shows that Klein does not investigate the affine group, and therefore also has not placed it yet between the main group and the group of collineations. Klein later explained this as the "consequence of one-sided traditions", which at the time had kept him from "fully appreciating the work done by Möbius and Graßmann"[200]. "Only in 1895/96 did Klein begin – in his lectures on number theory – to focus on the affine group as a set of linear substitutions of odd order, presupposing non-homogeneous variables."[201]

But Graßmann's influence did not remain limited to the Erlangen Program. We must also emphasize his work's importance in the axiomatic foundation of geometry. As early as 1844 Graßmann had felt the necessity for a theory giving a mathematical and abstract view of geometry, completely detached from the theory of real space and its properties, and had founded this theory in the "pure elements" ("Elemente schlechthin") of his *Extension Theory*. In 1889, Peano, in his *I principii di Geometria, logicamente expositi* (Peano 1889a), had rediscovered this idea by viewing basic geometrical elements as "things" to which the most diverse significations could be attached.[202] Hilbert, in his *Foundations of Geometry* (Hilbert 1900), completely and thoroughly developed this point of view in 1899.[203]

The Graßmannian cross ratio, the generation of third-order algebraic curves and surfaces (derived from a projective interpretation of planimetric and stereometric products), and the concept of the Graßmannian manifold[36] passed over into the repertoire of projective geometry. Severi's work established clearer contours for the concept of the Graßmannian manifold at the beginning of the 20th century.[204]

3. Abstract group theory

H. Wußing wrote the following words in his *The Genesis of the Abstract Group Concept,* a study in the history of mathematics: "But one may also justifiably consider 1882 as decisive for that [abstract group] concept's full elaboration. For in that year there appeared a paper that consciously combined all three of the concept's historical roots – the theory of algebraic equations, number theory, and geometry. … The decisive paper alluded to above was due to W. v. Dyck (1856 – 1934)."[205]

In his studies, W. v. Dyck, who as a pupil of Klein had learned about all three areas in which group theory was applied, relied heavily on Cayley's earlier work.[206] But when it came to his abstract concept of groups, he explicitly referred to Graßmann's *Extension Theory* and Hankel's *Theory of Complex Number Systems*, appreciating in retrospect both mathematicians' outstanding work in the field of abstract algebra and giving it a modern form. In the introduction to his 1882 *Group Theoretical Studies*, he wrote: "By viewing group theoretical operations in a purely formal way, we can clearly identify their role in the formal development of analytical operations. They are operations of multiplication which follow the associative, *but not* the commutative principle."[207] And in a footnote he added: "I will cite here Graßmann's reflections on the multiplication of extensive magnitudes in his *Extension Theory*, some sections on the multiplication of quaternions in Hamilton's *Elements of Quaternions*, and Hankel's lectures on complex numbers, … finally also the works of E. Schröder, who especially focuses on the place of group theoretical operations within the formal development of algebra."[208]

Evidently, Graßmann's works, dating back almost 40 years, still made an exceptional impact on the final architecture of group theory. And even 16 years later, in Whitehead's hands, Graßmann's algebraic and geometrical inquiries served as a basis for his revision of algebra and played a role in the mathematical concept of formalism.

Whitehead, in *A Treatise on Universal Algebra* (1898), on the influence of Graßmann on *Principles of Universal Algebra*

"The discussions of this chapter are largely based on the 'Uebersicht der allgemeinen Formenlehre' which forms the introductory chapter to Graßmann's Ausdehnungslehre von 1844."[209]

In his 1898 *A Treatise on Universal Algebra* (Whitehead 1898) Whitehead wanted "… to exhibit the algebras both as systems of symbolism, and also as engines for the investigation of the possibilities of thought and reasoning connected with the abstract general idea of space"[210]. Like Hankel before him, he was enthusiastic about the basic thoughts and the algebraic developments both *Extension Theories* presented. "The greatness of my obligations in this volume to Graßmann", he wrote in the introduction to his book, "will be understood by those who have mastered his two *Ausdehnungslehres*. The technical development of the subject is inspired chiefly by his work of 1862, but the underlying ideas follow the work of 1844."[211] Understandably, Whitehead only took up Graßmann's formal approaches;[212] but they served him in taking contemporary mathematics to a new level of abstraction. Graßmann's work, dating back half a century, played a major role in making the fundamental crisis in early 20th century mathematics more acute and thereby also represented a driving force in the further clarification of basic mathematical terminology.[213]

4. Vector and tensor calculus

Graßmann's foundational work in the field of vector and tensor calculus represents, apart from his elaboration of exterior algebra, one of his main scientific achievements in mathematics. But Graßmann shares the honor with Hamilton.[214]

Both mathematicians' investigations into extension theory and the theory of quaternions respectively, coincide in the fact that they deal directly with oriented magnitudes and only in a second step go on to a component representation. Both create a more general meaning for the word "product" and follow parallel paths on their way to ordinary vector calculus. While Graßmann's 1844 *Extension Theory* treats the theory of invariants of a group of affine transformations which leave the origin of coordinates unchanged, Hamilton's theory of quaternions and Graßmann's development from 1862 of the inner product focus on the theory of the rotation group in Euclidean space. From a terminological point of view, though, there are fundamental differences in the work of these two mathematicians.

The interplay of common features and differences in Graßmann's and Hamilton's thinking led to a fierce feud between two national "schools", which had begun to form around 1890, consisting of Graßmannians and Hamilton's Quaternionists. It lasted until the First World War, both parties claiming to be the only and true representatives of the respective school of thought.[215] The consequence of this was total conceptual chaos, which was only made worse by the Graßmannians' manic ambition of creating an ever new, specifically German terminology.[216] The independent lines of reception and application of Hamilton's and Graßmann's work by Gibbs (1881) in the USA and Peano (1888) in Italy only added to this confusion.[217]

Bourbaki on Peano's importance

"Peano, one of the creators of the axiomatic method, and also one of the first mathematicians to appreciate at its true worth the work of Graßmann [see Peano 1888, 1889a, 1898 – H.-J. P.], gives … the axiomatic definition of vector spaces (whether of finite dimension or not) over the field of reals, and, with fully modern notation, linear maps from such a space to another"[218]

Maxwell's *Treatise on Electricity and Magnetism* (Maxwell 1873) introduced vector calculus under Hamilton's terminology into British physics. American physics was under the sway of the Gibbsian synthesis of Hamilton's and Graßmann's terminology of vector calculus, and German physicists became familiar with Graßmann's and Hamilton's vector theories through Gibbs and through *Electromagnetic Theory* by the British telegraph engineer Heaviside (Heaviside 1951).[219] To this day, we must live with the divergent nomenclature these developments brought about.

Einstein's theory of relativity gave an enormous impulse to vector and tensor calculus at the beginning of the 20th century. The requirements posed by mathematical physics in dealing with four-dimensional space were decisive for their further elaboration. The further development of vector and tensor calculus in mathematics took place within the framework of linear algebra and, in the final decades of the last century, stimulated by the progress and demands of computer technology, took them to the abstract form they have today.

5. Exterior algebra

Graßmann's exterior algebra on a vector space of finite dimension over the field of real numbers was one of the first manifestations of associative, non-commutative algebra[220], which nevertheless did not become part of the repertoire of abstract algebra until after 1900.[221] Exterior algebra has become an important part of modern mathematics thanks to its many areas of application. Graßmann already understood its relevance in the theory of determinants, and Hankel continued this line of thought in 1867. By the end of the 19th century, a large number of publications dealt with the theory of determinants through the exterior product of *n* vectors within an *n*-dimensional vector space over the field of real numbers. Even our contemporary textbooks rely on such an approach for the theory of determinants.[222]

Exterior algebra over a Hilbert space of finite dimensions, together with the theory of determinants, gives important insights into the nature of the determinants of block matrices.[223]

The exterior product of tensors is yet another common feature in today's mathematical repertoire, an application of exterior algebra pointing back to Graßmann.[224] Also, modern literature has expanded the range of applications of exterior algebra to unitary R-right modules ("exterior power of a module"). The theory of associated subspaces of ideals in exterior algebra developed by E. Cartan should also be mentioned here.[225]

The theory of exterior differential forms, also going back directly to Graßmann, is exceptionally important and has also proven to be an application of exterior algebra. In 1862 Graßmann laid the foundations for this theory by approaching the Pfaffian differential equations with skew-symmetrical tensors. E. Cartan continued Graßmann's work, created a generalized form for his investigations and used the exterior differential forms to construct differential geometry.[226] Bourbaki emphasized that thereby Graßmann's work had finally received the place it deserves in the structure of mathematics.[227]

This goes to show how influential Graßmann's work was decades after its publication. Bearing this in mind, profound respect is the only adequate reaction to Graßmann's scientific achievements, which he attained living in the isolation of his hometown of Stettin and bearing the weight of his far-reaching obligations as a teacher and as an active member of society.

Notes

1 Clebsch 1871, p. 3.
2 See Wußing 1984, p. 25.
3 First steps towards an analytical geometry can be traced via E. Torricelli, G. Galilei, J. Kepler, N. Oresme to Appolonius of Perga. – See Böhm et. al. 1975, p. 16sqq.
4 See Struik 1987, 96sqq.
5 See Clebsch 1871, p. 10.
6 See Clebsch 1871, p. 10.
7 Leibniz 1686, p. 22.
8 In 1890 F. Engel noted that in analytical geometry something "unsteady and irregular" came to light, "for the single steps of the calculation almost never make any geometrical sense, they usually seem like a sleight of hand…" (Engel 1890, p. 17/18).
9 See Wußing 1984, p. 35sqq.
10 See part 4 of the present chapter.
11 See Study 1898, p. 159.
12 See Loria 1888, p. 115sqq.
13 See Alexandroff/Markuschewitsch/Chintschin 1971, p. 342sqq, and *History of science*, vol. 3 1965, p. 34sq.
14 It was Cayley who coined the term "*n*-dimensional geometry". – See Alexandroff/ Markuschewitsch/Chintschin 1971, p. 342sqq.
15 Lie 1934, p. 105.
16 See Licis 1976, p. 87sqq.
17 See Wußing 1976, p. 53sqq.
18 See Ruzavin 1977, p. 99sq., also Helmholtz 1868, Riemann 1876a, and Klein 1921.
19 See Blaschke 1948, p. 12sqq., and Wußing 1984, p. 27sq. These authors are also relevant in the following lines.
20 See Wußing/Arnold 1975, p. 270sqq.

21 Monge's groundbreaking work in differential geometry, which was continued notably by Gauß and Riemann and which shaped modern geometry, will not be analyzed in the present context.

22 See Fano 1907.

23 Wußing 1984, p. 26.

24 See ibidem.

25 Ibidem, p. 33.

26 In the present context, the remaining fundamental questions concerning the independent axiomatic foundation of geometry, which Hilbert, Poincaré, Pasch, Weyl and others tackled, will have to go unmentioned.

27 It should be emphasized here that we are not dealing with logical contradictions in this case.

28 Quoted from Wußing's introduction to Klein 1974, p. 20.

29 See Gerhardt 1877, p. 289sq., and Struik 1987, p. 165.

30 See chapter 1, section 4.

31 In other texts, the terms "multivector", "vector of the *n*th order", "homogeneous component of the *k*th order of an exterior vector", "*p*-vector", "completely alternating tensor", "purely co- or purely contra-variant tensor of the order *p*, alternating in all indexes" are used synonymously.

32 See also chapter 4, section 1.

33 EBBE, p. 11.

34 Quoted from Struik 1987, p. 135.

35 EBBE, p. 17.

36 More information on "Graßmann's or exterior algebra" and similar concepts from the viewpoint of recent mathematics can be found in: Naas/Schmid 1972a, p. 130, 648; Naas/Schmid 1972b, p. 209sq.; Kupcov 1977; Oniščik 1977a; Oniščik 1977b; Eisenreich 1971, p. 40sqq.; Gröbner 1966, p. 30sqq.; Groh 1956; Schatz 1970.

37 See EBBE, p. 19/20.

38 See ibidem, p. 28.

39 See ibidem, p. 30.

40 See ibidem, p. 40.

41 See ibidem p. 56sqq.

42 EBBE, p. 57.

43 EBBE, p. 81.

44 See Justus Graßmann's, Hermann Graßmann's son's, remark concerning the dissertation. In GW31, p. 221.

45 See chapter 1, section 4.

46 See section 4 of the present chapter.

47 In his equipollent calculus (Bellavitis 1835) Bellavitis developed a way of calculating with equipollent displacements, defining two displacements as equipollent when they are identical in length, direction and orientation, but not in situation. In closer examinations he found vector addition, among other insights. – See Rothe 1916.
48 See, for more details, Becker/Hofmann 1951, p. 326sqq.
49 Klein 1979, p. 161.
50 Biermann 1973, p. 38.
51 Wußing/Arnold 1975, p. 352.
52 See also Clebsch 1871, p. 15sq., and Clebsch 1874, p. 12.
53 See chapter 1, section 4.
54 Klein 1979, p. 162.
55 Clebsch 1871, p. 8.
56 See chapter 4.
57 See Graßmann's general elaborations in A1, p. 33 – 43.
58 A1, p. 40.
59 A1, p. 42.
60 Birjukova/Birjukov 1997 called attention to the fact that Graßmann, with his general theory of forms, was the first mathematician to present – in the context of a genetic-constructive mathematical program – an *abstract* theory of groupoids, half-groups, quasi-groups, groups, commutative groups and rings.
61 See A1, p. 45 and 154.
62 See chapter 4.
63 See A1, p. 48.
64 A1, p. 47.
65 F. Engel, the editor of Graßmann's collected works, makes this comment on the 1844 *Extension Theory* in GW11, p. 404.
66 A1, p. 50.
67 Riemann 1876a, p. 257. English translation taken from Smith 1929.
68 Riemann 1876a, p. 255.
69 A1, p. 58.
70 See Segre 1912, p. 22/23.
71 A1, p. 59.
72 A1, p. 60.
73 A1, p. 62.
74 Ibidem.
75 A1, p. 62/63.
76 See, among other sources, Helmholtz 1868.
77 Helmholtz 1868, p. 49.
78 Ibidem p. 32.

79 A1, p. 73. – See also my presentation of Graßmann's ideas concerning the essence of the mathematical method in chapter 4, section 4.

80 See chapter 4, section 3, for more details.

81 In this context, Klein speaks directly of the "Grassmann Determinant principle for the plane" and the "Grassmann principle for space". See Klein 1939, p. 21sqq., 29sqq.

82 See A1, p. 91 – 93.

83 See also section 8 of the first chapter.

84 See A1, 113sqq.

85 A1, 133/134.

86 Enriques 1907, p. 53.

87 Graßmann writes: "Instead of shadowing the result of a fundamental conjunction [vector addition, exterior multiplication, as well as the corresponding inverse operations – H.-J. P.], one can shadow its terms in the same sense." (A1, p. 144).

88 A1, p. 157.

89 A1, p. 173.

90 A2, p. 144.

91 See A1, p. 172 – 179.

92 A1, p. 186.

93 See Clebsch 1871, p. 28.

94 See Wußing 1984, p. 43.

95 See Clebsch 1871, p. 28 (footnote).

96 See A1, p. 197sq.

97 See also the explanations in Rothe 1916, p. 1284sqq.

98 Later, Graßmann became aware how unfruitful the level of extreme abstraction was when it came to introducing the regressive product. For this reason, when in 1877 he presented the second edition of the 1844 *Extension Theory*, he added an appendix to this section, in which he gave a simpler and more accessible definition of the regressive product. See A1, p. 295.

99 See A1, p. 202

100 A1, p. 207.

101 A1, p. 235.

102 Ibidem.

103 A1, p. 236.

104 See Graßmann's investigations in H. Graßmann 1851b.

105 See in this context Scheffer's remark in GW21, p. 393sqq.

106 Klein 1928, p. 132.

107 Ibidem.

108 See Bloch 1951.

109 In 1929, the theory of curves is occasionally mentioned in the work of the "Graßmann-adept" A. Lotze in Lotze 1929.
110 A1, p. 233.
111 These are the works by H. Graßmann 1846, 1848a, 1851a-c, 1852, 1855a-e, 1855g.
112 Cremona 1860, p. 356sq.
113 Möbius 1827, p. X.
114 A1, p. 248.
115 See Klein 1927, p. 10sqq.
116 See A1, p. 252.
117 Klein 1927, p. 11/12. This section of the text is not included in the English translation of Klein's *Development of Mathematics in the 19th Century* (Klein 1979).
118 A1, p. 255.
119 See A1, p. 252sq.
120 Concerning the Graßmann cross ratio, see Blaschke 1948, p. 95sq., and Keller 1963, p. 235sqq.
121 Möbius, who had worked intensively on similar problems, became aware of this connection, which he had used indirectly, through this work of Graßmann. – See Engel's remark concerning the 1844 *Extension Theory* in GW11, p. 411sqq.
122 See A1, p. 271.
123 The 1861 *Extension Theory* (A2) presents an extensive analysis of open products ("Lückenprodukte").
124 See A1, p. 273sqq.
125 Lotze 1929, p. 79sqq., is one example where the connection between Graßmann's open products and tensors is shown.
126 The original French passage from Leibniz' letter is reprinted in GW11, p. 417 – 420.
127 Leibniz in a letter to Christian Huygens (1629 – 1695) dated 8 September 1679. In: Leibniz 1976, p. 248 – 258 (p. 250). See also GW11, p. 418. A German translation of this passage, lacking bibliographical references, can be found in Bell 1967, p. 129.
128 PREIS, p. 318.
129 See Graßmann's letter to Hankel, 2 February 1867. The corresponding passage is reproduced in BIO, p. 110, footnote.
130 See also Engel's remark in GW11, p. 421.
131 PREIS, p. 320. So this amounts to saying that the following theorem is not universally valid: $(a\,c\,8\,b\,c) \wedge (a\,c\,d\,8\,a\,c\,d) \Longrightarrow (a\,c\,d\,8\,b\,c\,d)$.
132 See PREIS, p. 321.
133 PREIS, p. 332.
134 PREIS, p. 334.
135 PREIS, p. 333.
136 PREIS, p. 336.

137 PREIS, p. 339.
138 PREIS, p. 340.
139 See also Engel's remark in GW11, p. 421.
140 Here, Couturat's appreciative reaction to Graßmann's *Geometric Analysis* must be mentioned. – See Couturat 1969, p. 529 – 538.
141 See chapter 1, section 8.
142 A2, xiii/xiv
143 See Wußing's remarks in Wußing 1984, p. 239.
144 With these inquiries, Graßmann partially went beyond the results which Weierstraß and Jordan reached six and eight years later. See Bourbaki 1971, p. 109.
145 See chapter 4, sections 2 and 3.
146 Klein 1979, p. 165. In the original, the last word of this quote reads 'space', a translation error.
147 ZL, p. 3.
148 ZL, p. 4.
149 ZL, p. 6.
150 A1, p. 23.
151 A1, p. 24.
152 A1, p. 25.
153 A1, p. 23 (footnote).
154 LA, p. V.
155 Ibidem.
156 This is how A. Heintze remembers his days as a pupil of Justus Graßmann: "Also in mathematics we were given no problems in written form, and that wasn't good. We lost the habit of solving them, and when a paper had to be written, we mostly wrote it in an unorganized fashion, even copying from one another, which caused great trouble for Graßmann." (Heintze 1907, p. 44).
157 Ibidem.
158 See chapter 2, section 2.
159 LA, p. 3.
160 LA, p. 17.
161 LA, p. 21.
162 Interestingly, Helmholtz later turned to Graßmann's arithmetic in order to discuss under which conditions magnitudes in physics may be conceptualized as named numbers. He put the problem this way: "In which objective sense may we express the relations of real objects by named numbers as magnitudes, and under which conditions may we do so?" (Helmholtz 1887, p. 304).
163 See § 7 Division in: LA, p. 45sqq.

164 I will just remark that Graßmann did not develop the transition from rational magnitudes in the LA to the A2's real magnitudes.

165 The first of these two sentences is missing from Lloyd Kannenberg's translation of A2 (p. 3.). See also GW12, p. 12.

166 A2, p. 4.

167 R. Graßmann 1890c, S. VI.

168 This is also the case in Lewis 1995 and the noteworthy reflections by Radu 2000, p. 205.

169 Schlegel 1878, p. 42.

170 The following passages are analyzed in detail in Radu 2000, p. 205sqq.

171 See LA, p. 3.

172 See Hao Wang 1957, p. 147.

173 LA, p. 4.

174 Radu discusses at length that Graßmann's explanation (LA, p. 4): "$a + (b + e) = a + b + e$" is not constructively justified (2000, p. 216sqq).

175 When Schleiermacher remarked: "So the identity of the process and the immutability of the relation between thought and object are the two fundamental aspects of knowledge." (Schleiermacher 1942, p. 130), this probably was a strong reason for Graßmann to accept the validity of recursive proofs and the "pre-logical" characteristics of the complete induction's validity.

176 LA, p. 16. (Italics added.)

177 LA, p. 17.

178 LA, p. 18.

179 See LA, p. 19.

180 Hao Wang 1957, p. 148.

181 Hao Wang 1957, p. 147.

182 See also Radu 2000, p. 214.

183 LA, p. 3.

184 See R. Graßmann 1890f., p. 3 – 7.

185 See Zahn 1874, p. 583sqq.

186 See Bourbaki 1971, p. 33/34.

187 Hankel 1867, p. 1 – 34.

188 Wußing 1984, p. 240 (footnote 224).

189 See Bourbaki 1971, p. 33, 34, 71.

190 For more details, see Wußing 1984 and Bourbaki 1971.

191 See section 1 of this chapter.

192 Hankel 1867, p. 12.

193 Ibidem, p. 13.

194 Ibidem, p. 16.

195 Hao Wang emphasizes: "It is rather well-known, through Peano's own acknowledgement…, that Peano borrowed his axioms from Dedekind and made extensive use of Graßmann's work in his development of the axioms." (Hao Wang 1957, p. 145). – See also Wußing 1984, p. 239.
196 Letter from F. Klein to F. Engel, 21 January 1911. In: BIO, p. 312.
197 See Klein 1974, p. 55, 58sq.
198 See Klein 1974, p. 68.
199 See ibidem, p. 71.
200 Klein 1921, p. 320.
201 Wußing 1969, p. 143. See also Klein's thorough appreciation of Graßmann's work and his concomitant explication of affine and projective transformations in *Elementary Mathematics* ("Elementarmathematik", Klein 1925).
202 See Ruzavin 1977, p. 59, and Wußing 1984, p. 240 (footnote).
203 Toepell 1995 offers a detailed reconstruction of Graßmann's areas of influence in Hilbert's foundation of geometry.
204 See Severi 1916. See also Burau 1953.
205 Wußing 1984, p. 238.
206 See ibidem, p. 239sqq.
207 Dyck 1882, p. 2.
208 Ibidem. See also in Dyck 1882. p. 43 (footnote).
209 Whitehead 1898, p. 32.
210 Whitehead 1898, p. v.
211 Ibidem, p. x.
212 See Natorp 1901 for a description of Whitehead's formalistic approach to mathematics.
213 In the context of the theory of algorithms, Birjukova and Birjukov (1997) point out that Graßmann's "general theory of forms" (A1 1844) already gave a definition for the concept of the abstract group (10 years before Cayley) and of the ring (70 years before Fraenkel). This fact has so far gone unnoticed.
214 See in this context M. J. Crowe's detailed analysis in *A History of Vector Analysis* (1994).
215 Concerning the "Graßmannians", see Schubring 1996b. Crowe 1994 offers a statistical analysis of works published by the followers of Graßmann's and Hamilton's.
216 See the terminology chosen by A. Lotze in Lotze 1929. Burali-Forti 1921, p. 239 – 244, gives an instructive overview of the different concepts used up until 1920.
217 Polak, in his Hamilton-biography 1993, p. 233 – 236, tries to show that Peano, in his *Calcolo geometrico secondo l'*Ausdehnungslehre *di H. Graßmann* (1888), does not actually follow Graßmann, but Gibbs, who had composed two small treatises on the *Elements of vector analysis* in 1881 and 1884, passing on to friends, among them G. Basso, Peano's teacher, and J. Lüroth, a friend of Peano, 130 exemplars of these texts. This amounts to saying that it was Gibbs, who according to Polak had developed vector algebra mostly on his own,

is at the beginning of the modern theory of affine vector spaces and that Graßmann had no real effect in this sense (see the overview in Zaddach 1994, p. 11). But Polak's argument, which relies especially on the clarity of Peano's thinking and Graßmann's obscurity in the corresponding passages, is not a convincing one. See in this context the Peano-biography by Kennedy 2002.

218 Bourbaki 1998, p. 66.

219 See Klein 1939, p. 51sqq.

220 Clifford algebra must be considered a generalized form of Graßmannian algebra.

221 See also Bourbaki 1961, p. 140.

222 See, as an example, Gröbner 1966, p. 7.

223 See Pillis 1968.

224 See for example Eisenreich 1971, p. 132sqq.

225 See in this context Groh 1956.

226 See Cartan 1952, p. 38/39, and Cartan 1953, p. 241/242.

227 See Bourbaki 1971, p. 83sq.

4 The genesis and essence of Hermann Günther Graßmann's philosophical views in the *Extension Theory* of 1844

Hermann Graßmann's philosophical and methodological views were directly on the path his father had traced for him. His father had been influenced by Pestalozzi's pedagogy, Leibniz' combinatorial and synthetic approach, Kant's constructive view of mathematics and the dialectics of Romantic philosophy of nature. Merged together, these were the elements of Justus Graßmann's unique philosophical and mathematical position. These impulses – modified by Schleiermacher's dialectics – also shaped Hermann Graßmann's way of thinking.

Leibniz' and Graßmann's kinship in spirit

"Therefore, just as the other sciences have to access certainty following the example of mathematics, so too the asperity of mathematics must be mitigated by a softer mode of proceeding {blandior tractandi ratio} that follows the example of the other sciences … For the same reason, demonstrations should also be performed without algebraic calculus. Even though algebra is very useful and I appreciate it very much, and it is needed for results that otherwise we could not obtain, one should nevertheless avoid using it whenever a truth can be proved by a certain natural reason that guides the mind through the very ideas of things."[1]

In his *Extension Theory*, Hermann Günther Graßmann developed basic positions for a – in his day – quite workable philosophical and methodological foundation of mathematics, which proved its worth in his mathematical achievements and went far beyond what contemporary idealist philosophy had to offer. His attempt at purposefully employing dialectics in order to solve fundamental questions of philosophy and the theory of science is especially noteworthy.

4.1 The genesis of *Extension Theory's* basic principles

In 1844 Hermann Graßmann had become aware that he had created a new mathematical theory. Following his father's example, who had felt obliged to explain the genesis of his ideas when he published his newly established *Geometric Theory of Combinations* ("Geometrische Combinationslehre", KRY) in 1829, Hermann Graßmann also wanted to tell the readers of his *Extension Theory* how he had been "led, step by step, to the results presented here"[2].

These explanations give us, along with a letter to Saint-Venant[3], a relatively complete overview of the different phases in his elaboration of vector algebra and affine geometry.

According to Graßmann himself, "the consideration of negatives in geometry" provided the "initial incentive"[4] for developing vector algebra. Immediately one is reminded of one of Kant's early writings, dating from 1763 and entitled *Attempt to Introduce the Concept of Negative Magnitude into Philosophy* (Kant 1912b). However, we do not know to what extent this text, which generalizes the concept of the negative as a concept of dialectical opposition, had an influence on Graßmann's thoughts.

From 1830 to 1832, after finishing his university studies, Graßmann had begun to learn about the essential structures of mathematics. Relying on his father's textbooks and manuscripts, he simultaneously studied arithmetic, algebra and geometry. He summarized the first results of his inquiries in a small treatise, which regrettably has been lost: *On Geometrical Analysis and the Application of Arithmetic and Algebra to Geometry* ("Über die geometrische Analyse und über die Anwendung der Arithmetik und Algebra auf die Geometrie")[5].

His particular methods of study, which were inspired by the mode of presentation of his father's writings[6], persuaded him, at an early stage and following his father's approach, to make an attempt at connecting algebra and geometry in a new way, deviating from Descartes. This is confirmed by statements of Graßmann according to which, as early as 1832, he had found the first elements of his vector algebraic calculus, had obtained vector addition and the exterior multiplication of vectors.[7] In the foreword to his *Extension Theory*, he informed the reader about the separate steps he had gone through in elaborating his calculus.

Considering "negatives in geometry" took him to "regarding the displacements AB and BA as opposite magnitudes. From this it follows that if A, B, C are points of a straight line, then $AB + BC = AC$ is always true…"[8] But by formally transferring an arithmetical operation to geometrical relationships, Graßmann was confronted with an inequality between the sum of the *displacements' lengths* and the displacements' sum when their *direction* was taken into account. By maintaining the formal validity of the laws of addition, Graßmann obtained a generalized form of geometrical addition for displacements with any given direction! "This can most easily be accomplished", he wrote, "if the law $AB + BC = AC$ is imposed even when A, B, C do not lie on a single straight line. Thus the first step was taken toward an analysis that subsequently led to the new branch of mathematics presented here."[9]

This is how Graßmann managed to uncover new structures in geometry by making the conceptual differentiation ($AB \neq BA$), by transferring concepts from algebra to geometry ("addition"), by extending these concepts' area of validity (the displacements which are to be added may be oriented in different directions) and by requiring permanence for the operations established so far ($AB + BC = AC$ must be valid for all given displacements). He thereby also created a new synthesis of algebra and geometry.

After having established vector addition, the transfer of the algebraic concept of the *product* to geometrical operations with displacements was a second direct step towards establishing extension theory. Again, a formal expression of his father's was Graßmann's point of departure. In his *Plane Spatial Theory of Magnitudes* ("Ebene räumliche Größenlehre", J. Graßmann 1824), he had defined a rectangle's surface area as the product of two of the rectangle's adjacent sides.[10] Hermann Graßmann took up this idea and generalized the concept of the product for parallelograms, following the path he had already traced for vector addition, saying that they "may be regarded as products of an adjacent pair of their sides, provided one again interprets the product, not as the product of their lengths, but as that of the two displacements with their directions taken into account"[11].

By combining different characteristics of a geometrical object to form a new magnitude, that is, analyzing displacements and simultaneously taking into account their orientation and length, Graßmann had been led to a more complex geometrical entity with new structural properties. But these magnitudes' increase in complexity also simplified and abbreviated the mathematical instruments of representation, thereby rendering it possible to make the deeper mathematical structures transparent.

In the following lines, Graßmann "combined this concept of the product with that previously established for the sum"[12]. The relationships resulting from this thought showed, according to Graßmann, "the most striking harmony"[13] when compared to the addition and multiplication of numbers. But he was "initially perplexed by the remarkable result that, although the laws of ordinary multiplication, including the relation of multiplication to addition, remained valid for this new type of product, one could only interchange factors if one simultaneously changed the sign…"[14]

Graßmann's basic idea is so simple that it is easy to overlook the genius of his approach. Bell, in turn, emphasized the following point strongly by focusing on Hamilton's analogous investigations: "That a consistent, practically useful system of algebra could be constructed in defiance of the commutative law of multiplication was a discovery of the first order, comparable, perhaps, to the conception of non-Euclidean geometry."[15] Hamilton only reached this insight after fifteen years of fruitless reflections.[16]

Initially, Graßmann also was irritated by the violation of commutativity, which was a consequence of the inner logic of this approach. But, from his father's writings, he certainly must have been aware of cases of non-commutative conjunctions, which presumably laid his doubts to rest.[17]

Graßmann's vector algebraic ideas had taken him into the intellectual realm of the famous philosopher and mathematician Leibniz. Graßmann's most important predecessor 150 years earlier, Leibniz had also deviated from the Cartesian path in applying algebra to geometry. But Graßmann was the one to give a fruitful meaning to the concrete development of this new approach.

The further development of Graßmann's vector algebraic endeavors came to a standstill in the years after 1832 because, as he put it, "the demands of my job led me to other tasks".[18] Only his 1840 graduation thesis on the theory of tides (EBBE) would take him "to Lagrange's *Mécanique Analytique*, and thence back to the ideas of this analysis. All the developments in that work were transformed by the principles of this new analysis into such simple procedures that the calculations often came out one-tenth as long as there."[19]

When Graßmann pointed to his professional obligations to explain his lack of progress in the further development of his geometrical analysis, this seems quite unconvincing when we take his work on the theory of tides into account. Firstly, the exceptionally heavy burden of further occupations[20] did not hinder him from putting the latter piece of work to paper. Secondly, in 1832 he obviously still had not realized how much mathematical progress one could expect from his theoretical approach.[21] Only when his vector algebraic method had proven itself in a problem of theoretical mechanics – the theory of tides –, Graßmann became completely convinced of its far-reaching importance. As often in the history of mathematics, a problem in physics helped to bring a mathematical theory to light.

H. Poincaré on the usefulness of physics for the mathematician

"In the first place the physicist sets us problems whose solution he expects of us. But in proposing them to us, he has largely paid us in advance for the service we shall render him, if we solve them. ...
History proves that physics has not only forced us to choose among problems which came in a crowd; it has imposed upon us such as we should without it never have dreamed of. However varied may be the imagination of man, nature is still a thousand times richer. To follow her we must take ways we have neglected, and these paths lead us often to summits whence we discover new countries. What could be more useful!"[22]

At the same time, the inner contradictions of Descartes' analytical geometry, which appeared strongly in Laplace's and Lagrange's modes of presentation, became the actual reason why Graßmann renewed his investigations from 1832 and, this time, really put them to work.[23]

In Graßmann's view, the direct translatability of mechanical relationships into the language of vector algebra was mirrored by the fact that "each step from one formula to another appears at once as just the symbolic expression of a parallel act of abstract reasoning."[24] Along with his father's conviction that mathematical results, should they be a "perfect representation of the synthesis achieved in the mind", would have to be "closer to the synthesis which nature achieves in her creations…"[25], this point of view enforced his will to continue on the path of his theoretical investigations.

His newly-found interest in vector algebra and affine geometry led Graßmann to investigate the affine point space. Initially limited to three-dimensional space, he transferred the concepts of addition and multiplication to operations concerning points.[26] In the course of these investigations, he discovered that his way of constructing the center of gravity in a system of mass points through the geometrical sum of the points matched Möbius' developments in his *Barycentric Calculus* (Möbius 1827). Additionally, this like-mindedness in dealing with geometry must have become visible to Graßmann in at least two further points of contact with Möbius:

Firstly, both scientists shared the same basic principle in operating with oriented displacements. Möbius wrote: "I will only remark that, throughout the book, I have used … and extended the known practice of expressing the positive or negative values of a line by different ways of switching around the letters which designate the end points of the line. Hereby, … the synthetic method's intuitive clarity is connected as closely as possible to the analytic method's generality by using purely geometrical symbols, the letters chosen to designate the points of a figure, to represent the arithmetical relationships between the parts of the figure in formulas which are valid for all possible positions of the parts."[27]

Secondly, in Möbius we can also find (though in a less developed state) Graßmann's idea of operating directly with geometrical objects in an algebraic way. In this sense, Möbius wrote: "But our formula [a centroid equation of three points – H.-J. P.] is more than a mere abbreviation of this theorem … this formula simultaneously [shows us] a central characteristic of the centroid in the language of algebra and, for this reason, we can work with it just like with any other algebraic equation. … I have done the calculation with such abbreviated formulas from what I have called the *barycentric* calculus, that is to say, the calculus derived from the concept of the center of gravity. This is a calculus which not only has to do with real number magnitudes, but apparently also with mere points, while, on the whole, it shows no difference compared to the common way of calculating in algebra."[28]

Understandably, Graßmann was enthusiastic about Möbius' *Barycentric Calculus*, the only contemporary piece of work on the European continent which came close to his own ideas. Nevertheless he had to recognize that Möbius had not chosen the path of generating products in geometry. This was the decisive reason for Graßmann to present his findings *systematically* and to edit them for publication. "Thus when I proceeded", he remarked, "to work out my results consistently and *from the beginning*, being careful *to appeal to no principle proven in any other branch of mathematics*, I found that the analysis I had discovered did not touch only on the subject of geometry, as it had seemed before. Rather, I soon realized that I had come upon the domain of a new science, of which geometry itself is only a special application." [italics added – H.-J. P.][29]

Graßmann's path of intellectual development is the main reason for his striving, discernible in the quotation above, for a closed systematic elaboration of his theoretical views, possessing as its underlying principles the greatest possible "purity" and the highest level of conceptual generality. On the one hand, we must remember the late awakening of Graßmann's interest in mathematics and his prior philosophical education, which had revolved around Schleiermacher and his philosophy. On the other, we will have to think of the intense influence his father's mathematical-philosophical conceptions exerted upon Graßmann. His father's methodological principles on how to approach the theory of number and his *Geometrical Theory of Combinations* were preliminaries to Hermann Graßmann's work.[30]

By developing extension theory "without preconditions" and systematically, Graßmann gained enormous mathematical insight. This way of dealing with the matter bore fruit by yielding, among other aspects, the foundation of a general theory of conjunctions (Graßmann's "general theory of forms") and the generalization of the traditional concept of geometry. As Graßmann emphasized, it is only by investigating *n*-dimensional spaces that "the laws come to light in their full clarity and generality, and their essential interrelationships are revealed; and many regularities, which for three dimensions either do not appear at all, or at best obscurely, present themselves perfectly clearly with this generalization."[31]

But this theoretical improvement meant that communication with the mathematicians of his time had become almost impossible: before one could appreciate the importance of Graßmann's theory, which went far beyond the contemporary concept of mathematics, one had to have understood his entire *system*.[32] Graßmann's hope that his theory might become "a living limb of the organism of science"[33] could hardly fulfill itself in 1844.

In Graßmann's opinion, the systematic development of a new mathematical discipline had to include a comprehensive conception of the world and of a methodology. Here he was ahead of many contemporary mathematicians[34]: "Indeed, for the position and significance of a new science to be properly recognized", he wrote in the foreword to his *Extension Theory* of 1844, "it is absolutely necessary to show its applications and its links to related subjects simultaneously with its presentation. The introduction serves this purpose as well. It presents the nature of the subject from a more philosophical viewpoint..."[35]

Since mathematicians and scientists thought very little of Hegelian philosophy, Graßmann could foresee that his dialectical considerations would hardly be appreciated and therefore he drew a line: "For there prevails among mathematicians (and not without some justification) a certain aversion to the philosophical discussion of mathematical and physical subjects; and in fact most analyses of this type, as for example those of Hegel and his school, permit an obscurity and arbitrariness that nullifies any fruits of such an analysis."[36]

Nevertheless, in the introduction to his *Extension Theory* of 1844, he developed and systematized his conceptions of a philosophical foundation of mathematics without letting the possible rejection of his theories by the mathematical community dissuade him.

4.2 Hermann Graßmann's basic philosophical principles concerning his determination of the essence of mathematics

By committing himself to creating a strict and systematic foundation for affine geometry and vector algebra, Graßmann had been led to a theory of n-dimensional manifolds which at the time neither had any correspondence to reality, nor any conformity with the traditional mathematical understanding of geometry. He had discovered a type of geometry which did not offer a description of physical space. Nevertheless, as Georg Klaus remarked, these types of geometry depend on the mental activity of human beings, given the fact that they are *constructed* by mathematicians.[37]

In reflecting philosophically on this, Graßmann inevitably had to deal with the problem concerning the ontological status of mathematics and found it necessary to determine the essence of mathematics in some fundamental way. "The principal division of the sciences", he began his philosophical considerations, "is into the real and the formal. The real represent the existent in thought as existing independently of thought, and their truth consists in the correspondence of the thought with that existent. The formal on the other hand have as their object what has been produced by thought alone, and their truth consists in the correspondence between the thought processes themselves."[38] According to Graßmann, all empirical sciences were subsumed under the concept of real sciences; he viewed dialectics and pure mathematics as being part of the formal sciences.[39]

The basic standpoint informing this quotation requires a closer analysis of this worldview because its philosophical content and historical relevance only become discernible against the backdrop of Graßmann's intellectual development.

In the first place, we will have to note that Graßmann never brought up the concept of *God* in his philosophical exploration of mathematics, either in this, or in any other passage. Here, he deviated from his father's writings on mathematics and the natural sciences, despite his theological education.[40] The strict theoretical development of his ideas left no room for an explicit religious “confession”. If we take into account that important mathematicians of his time, such as Bolzano and Cantor, were still incapable of presenting their philosophical arguments without introducing the concept of God and without relying on the teachings of the Fathers of the Church, we can fathom the greatness of Graßmann’s scientific realism.[41]

Graßmann’s almost complete separation of religion and science must largely be attributed to the influence the pantheistic elements of Schleiermacher’s philosophy exerted upon him. At an early stage in his *On Religion* (Schleiermacher 1913), Schleiermacher had demanded that religion be kept separate from philosophy and morality, and later, in his *Dialectic* (DIAL), he defined the concept of God as the limiting concept for the ultimate ground of existence, unreachable by scientific reflection.[42]

Schleiermacher (DIAL) on the concept of knowledge:

§ 14: “When one wants to grasp the rules of conjunction [of concepts in thought – H.-J. P.] in a scientific way, then they cannot be separated from the deepest foundations of knowledge. For in order to conjoin correctly, one cannot deviate from the way in which things are really conjoined, and the relationship between our knowledge and the things is our only proof.”[43]

§ 94: “In every thought something that is being thought is posited outside thought.”[44]

§ 96: “Every thought that refers to something that is posited outside it, but does not correspond to the thing that is posited outside it, is not knowledge.”[45]

§ 101: Knowledge is “the correspondence of thought to an existent”.[46]

Lecture of 1818: “For in every thought we posit something outside of thought that is being thought. What is being thought may be inside or outside us, but a state of mind [Zustand] and action within us still are not the same as thought, for both can exist without being thought. So the object, even though it may be an internal one, is outside thought and it is only within us not because we think, but because we exist.”[47]

Lecture of 1831: “*Knowledge = That kind of thought which corresponds to an existent*.”[48]

Graßmann linked his philosophical explanations directly to Schleiermacher’s views. According to Graßmann himself, he “owed [Schleiermacher] infinitely in matters of the spirit”.[49] A case in point is his concept of truth, cited above, – the correspondence of thought to an existent and of thought processes among themselves – which he took directly from the first paragraph of Schleiermacher’s *Dialectic*. It is important to note that Graßmann followed Schleiermacher in his “secular” approach to the concept of knowledge.

In the end, Graßmann’s realism, which manifested itself when he stated that the object of empirical science is an existent, “independent, existing for itself outside of thought" and "portrayed by the thought"[50], as in Schleiermacher, remained ambiguous because the existent's status was not determined more closely. But, if we recall the context of his biographical background-knowledge, the remaining question concerning the "ultimate ground of existence" must have had an answer for Graßmann.

This helps to explain why in a late treatise on the loss of faith (H. Graßmann 1878) he opted for theism and renounced the materialistic tendencies in science, which, in turn, he himself had implicitly strengthened in his *Extension Theory* by removing the concept of God from the philosophical foundation of mathematics.

***On the Loss of Faith. Appeals to the Scientifically Educated in Modern Times* (Hermann Graßmann 1878)**

"For decades now, the loss of Christian faith has become ever more widespread among educated people, hailed enthusiastically by some as a sign of progress, bewailed by others as the ultimate struggle against all religious, moral, spiritual life."[51]
"In such times", Graßmann went on to say, "it is our duty to remain firm and true to our faith, not to recoil from the thrust of the great, unfaithful multitude; we must acknowledge our faith, without hesitation, without fear, with total frankness; but we must also delve more deeply and purely into God's revelation ... we must, while holding on firmly to a strict and clear manifestation of our faith, also hold on to our spiritual unity..."[52]
Against the claims voiced by "unfaithful science", Graßmann especially defended his unshakable faith in miracles. Science supposedly claimed that miracles had to violate the laws created by God himself, something a God could not possibly permit:
"And this weak-minded conclusion is reiterated ... even in most recent times ... If one bears in mind how little we know about the laws of nature, how complete our ignorance is when it comes to the laws of plant life or even the soul-life of animals, and what is more, concerning the development of the world and the laws it obeys, how could we, paying no attention to our ignorance, draw such a shaky conclusion from such a short period of time. One will have to say that this proof against miracles really relies on an almost miraculous presumptuousness."[53]

Graßmann's subsequent thoughts in the introduction to his *Extension Theory* are impressively concise and strict in the way they develop his unitary basic conception.

After having divided the sciences into the formal and the real, he goes on to determine the formal sciences more precisely, following Schleiermacher closely. In his view, the formal sciences fall into two categories. They "treat either the *general* laws of thought or the *particular* as established by means of thought, the former being the dialectic (logic), the latter, pure mathematics."[54] And he goes on to say: Dialectics "is a philosophical science, since it seeks the unity in all thought, while mathematics has the opposite orientation in that it regards each individual thought as particular."[55]

The essence of Schleiermacher's *Dialectic:* the last 3 paragraphs

"§ 344. The idea of knowledge under the isolated form of the general is dialectics.
§ 345. The idea of knowledge under the isolated form of the particular is mathematics.
§ 346. Every real thought contains as much science as it contains dialectics and mathematics."[56]

As a consequence, Graßmann, following Schleiermacher[57], viewed dialectics primarily as a subjective dialectics and placed pure mathematics in the realm of ideal constructions. Nevertheless, Graßmann deviated from Schleiermacher when he left the question concerning the relation between the "real" and the "formal" sciences unanswered – that is, the relation between objective and subjective dialectics on the one hand and mathematics and reality on the other. Instead, Graßmann limited himself to a strictly structural analysis of contemporary science and, as a consequence, did not aim to explain the genesis of scientific thinking.

But it is very remarkable that he accorded such a fundamental place to dialectics in the system of the sciences, though he did not view it as an evolutionary dialectics[58], at a time when the Hegelian dialectics had become worthless in the eyes of mathematicians and natural scientists.[59]

Graßmann then developed his philosophical determination of the object of mathematics, building it upon the concept of truth which he had postulated before. When he stated that "thought exists only in reference to an existent that confronts it and is portrayed by the thought"[60], this was to say that, after mathematics had been placed among the "formal sciences", its object may only be taken as "established by thought itself, when a second thought process is confronted as an existent."[61]

By assuming this point of view, pure mathematics became, for Graßmann, "the science of the *particular* existent that has *come to be* by thought. The particular existent, viewed in this sense, we call a thought form or simply a *form*; thus pure mathematics is the *theory of forms*."[62] And he went on to explain: "On the other hand the expression 'form' might seem rather too broad, and the name 'thought form' more appropriate; but the form in its pure meaning, devoid of all content, is precisely nothing but the thought form…"[63]

In the further course of the book, Graßmann used this agenda, sketched in extremely abstract terminology and representing a *constructive* foundation of mathematics, to determine the objects of the different mathematical disciplines and to develop the basic concepts of his *Extension Theory*.[64] Before dealing with related questions, we will briefly turn to the intellectual roots of this conception.

The idea of a constructive foundation for mathematics goes back to Kant. In his *Critique of Pure Reason* (Kant 2007), in which he attempted to conceive recognition as the active process of entering into the essence of entities, mathematics and philosophy played an exceptional role as the two forms of thinking in which reason yields theoretical knowledge.[65] In his attempt at founding a theory of "synthetic judgments *a priori*", Kant primarily turned to mathematics. His answers to the ensuing questions, asking under which preconditions pure mathematics may exist, on the one hand, and what the essence of mathematics may be, on the other, were quite different. While his reply to the first question – How is mathematics possible? – led to subjective idealism, which in Kant's elaborations on transcendental esthetics manifested itself in his denial of the objectivity of space and time[66], he solved the second question concerning the essence of mathematics – What is mathematics? – by bringing forward the brilliant notion of a mathematical and constructive foundation.[67] The following quotation will just serve as one typical example: "*Philosophical* knowledge is the *knowledge gained by reason from concepts*; mathematical knowledge is the knowledge gained by reason from the *construction* of concepts. ... Thus philosophical knowledge considers the particular only in the universal, mathematical knowledge the universal in the particular, or even in the single instance, though still always a priori and by means of reason. Accordingly, just as this single object is determined by certain universal conditions of construction, so the object of the concept, to which the single object corresponds merely as its schema, must likewise be thought as universally determined."[68] With this point of view, Kant became the point of departure for a movement working on a foundation for mathematics, constituted in the late 19th century and leading into the present along the lines of a constructive approach to mathematics and the works of Brouwer, Weyl and Skolem.[69]

Hermann Graßmann adopted the Kantian view in a modified way from Schleiermacher's works and, most notably, from his father's writings.

In his treatise *On the Concept and Extent of the Pure Theory of Number* ("Ueber den Begriff und Umfang der reinen Zahlenlehre", ZL), Justus Graßmann had stripped the Kantian view of mathematics of its subjective and idealist dimensions. Following Kant, he located the essence of mathematics in the generation of "its first concepts through its own inherent synthesis (which we call a construction in the widest sense)"[70], while this synthesis "completely ignores the content of the elements that will be conjoined ... positing the conjoined elements as devoid of content"[71]. But, for him, the object of mathematics was "not the form of this synthesis, but the product resulting from it..."[72]

So J. Graßmann drew a line between himself and Kant when he stated that "no concept of truth may be attributed to this mathematical synthesis, and it is just this which distinguishes it from synthetic judgments..."[73] In J. Graßmann's view, contents in mathematics were rooted in the fact that its theorems were "statements about the characteristics of this specific mathematical synthesis, and about that which is simultaneously given along with it"[74].

Friedrich Schleiermacher, in turn, assumed that true thought could only exist when it referred to an existent. This explains why such an existent also had to be present in the pure (that is, scientific) thinking of mathematics, in the synthesis taking place in cognition. With the characteristics of knowledge in mind ("Identity of the process and immutability of the thoughts in their relation to the existent"[75]), the object of mathematics still remained undefined. For Schleiermacher, finally, the object of pure mathematics was the "action of the subject", and he remarked: "So action and being, as objects of thought, behave in exactly the same way."[76]

Schleiermacher also stated that the process of reflection in mathematics only begins with knowledge, and went on to clarify: "Namely, since we agreed on a concept called "thought", we cannot exclude the [mathematical – H.-J. P.] operations from thought. Nevertheless, because every one of these acts begins with total certainty, they constitute a domain completely closed in itself. … But it [mathematical thought – H.-J. P.] draws its certainty from the fact that it is an activity. Certainty only refers to this activity because the manifestation of this activity, as a symbol – that is to say, calculus –, is set aside as something insignificant. As a domain completely closed in itself, it contrasts with pure thought, which refers to an existent, and with everyday thought … As soon as it … becomes active in all its applications … it becomes aware of its uncertainty concerning its relation to an existent…"[77] The elements of a constructive, "operational" understanding of the essence of mathematics appear very clearly, and the problem of the application of mathematics is already visible here.[78]

Hermann Graßmann took up this approach of a constructive foundation for mathematics, which had started with Kant, had been modified by Justus Graßmann and enriched by Schleiermacher's conception of the essence of mathematics. It appeared in his characterization of mathematics as a "theory of forms", preceding his *Extension Theory*, and became the core element of his subsequent philosophical and mathematical investigations. Hereby he became the precursor of a mathematical movement which only began to form by the late 19th century, in connection with the fundamental crisis of mathematics, and which – in its modern shape – gave important mathematical results.

The definition of the object of mathematics as "the *particular* as established by means of thought"[79] went along with another element, which simultaneously extended and surpassed the traditional understanding of mathematics.

In Kant, the constructive generation of mathematical objects had remained linked to the concept of quantity: "The form of mathematical knowledge is the cause why it is limited exclusively to quantities. For it is the concept of quantities that allows of being constructed…"[80] Up until the mid-19th century, mathematics was considered the science of quantities and their relationships.

But for Hermann Graßmann this traditional characterization of the object of mathematics had already become too limited, since he, just like his father, took the theory of combinations to be an essential branch of mathematics, while permutations did not appear as magnitudes. So it was strictly logical for him to emphasize in his *Extension*

Theory: "The name 'theory of magnitude' is inappropriate for all of mathematics, since one finds no use for magnitude in a substantial branch of it, namely combination theory..."[81] This is why he extended the concept of mathematics to include the theory of forms. Next to Bolzano and Boole[82], Graßmann paved the way for our modern, abstract understanding of mathematics, which views mathematics[83] "as a storehouse of abstract forms – the mathematical structures"[84].

The following discussion of Graßmann's concept of the element illustrates the extraordinary level of abstraction which he attained in his foundation of mathematics, and which brings him into the vicinity of Cantor, the founder of set theory: "First, in place of the point ... we here substitute the *element*, by which we mean a simple particular, conceived as distinguished from other particulars; and indeed we attribute to the element in the abstract science absolutely no other content. There can therefore be no question as to what sort of particular it actually is – for it is simply a particular, devoid of any real content –, or in what sense this one is distinguished from the others – for it is merely defined as distinct, without establishing any real content to the sense in which the distinction exists."[85]

So even though the object of mathematics is in this sense grasped as "simply a particular", mathematics is always viewed as part of the universal, that is to say, it participates in the potential possibilities of being interpreted by a diversity of relationships in reality. Seeing extension theory as a mathematical science, developed from geometry by means of abstractions, Graßmann showed how this potentiality could be realized also in cases of *non-spatial* relationships, such as the mathematical treatment of the theory of color. Here, as he demonstrated, each color valency could be represented by a three-dimensional vector of color valencies, with colors acting as representatives of non-spatial extensive magnitudes.[86]

Robert Graßmann's opinion on Kant's philosophy also highlighted the general differences: "Kant further claims", he wrote, "that two forms of perception are *a priori* (not taken from experience, but necessary intuitions *a priori*) to mathematics: namely that the *a priori* intuitions of time, for numbers, and of space, for the theory of space, are foundational. This claim also follows from an error of Kant's. In rigorous mathematics, as it is developed in the theory of cognition, there are no underlying forms of perception."[87]

Summing up, we get the following picture of Hermann Graßmann's views on the foundation of mathematics: Science aims to uncover truth. But since truth consists in a thought-content's correspondence to an independent entity, not arising from thought, thought as a process of establishing correspondence must always be confronted by an existent which is independent from it. But mathematical thought, in turn, which has no immediate reference to knowledge about the outside world, can only gain its object by an original act of ideal construction. Pure mathematics begins constructively by positing the simplest forms, which are abstracted from any real content. Then, in another cognitive step, the mathematician is confronted with these forms as givens, which he reflects upon in thought, connecting and reorganizing them.

The certainty of mathematical insights is based on the laws of logic; the mathematician ascends to ever more complex and differentiated mathematical relations by proceeding on the basis of the general dialectical rules of thought.

Intellectual activity creates the connections between mathematical forms, so mathematical diversity arises from "movement" in thought. The dialectical patterns of thought hereby determine the mathematical developments. In the course of its theoretical development, the particular, posited by thought, continuously gains more attributes, and it therefore becomes increasingly concrete on an abstract level and only maintains its particularity when it is detached from its empirical content.

As we have already seen, this conception of mathematics contains many elements which are highly relevant for today's discussions.

Because of the high level of abstraction in his philosophical and mathematical views, Graßmann, in his analysis of the foundations of mathematics, was capable of acceding to a new mathematical understanding of geometry. It was becoming increasingly clear to him, he wrote, "that geometry can in no way be regarded as a branch of mathematics like arithmetic or combination theory; instead, geometry relates to something already given in nature, namely space. I had also realized that there must be a branch of mathematics that yields in a purely abstract way laws similar to those that in geometry seem bound to space."[88]

While his father had viewed the object of mathematics as a given without any content whatsoever and nevertheless had inconsistently made geometry a part of mathematics[89], Hermann Graßmann removed geometry from pure mathematics because he did not consider physical space a product of consciousness. He created, in its place, abstract *n*-dimensional extension theory. Simultaneously, he formulated a critique of Kant's subjective-idealistic concept of space, highly remarkable at the time: "This is clear because the concept of space can in no way be produced by thought, but rather emerges as something given. Anyone who would maintain the contrary must undertake the task of deducing the necessity for the three dimensions of space from the laws of pure thought, a problem whose solution is patently impossible."[90]

Since for Graßmann mathematics was not founded on perception, but on thought, he could still tolerate the Kantian aprioristic conception of spatial perception because there was no reason to discard it as a psychological phenomenon: Only a few lines after having characterized objective space, Graßmann discussed the concept of human spatial perception. He explained: "Although just now we said that perception confronts thought as something independently given, it is not thereby asserted that space perception emerges only from the consideration of *solid objects*; rather, it is that fundamental perception *imparted to us by the openness of our senses to the sensible world*, which adheres to us as closely as body to soul. [italics added – H.-J. P.]."[91]

Only his brother would radically contradict Kant later: "It is also one of Kant's errors when he states: space and time do not arise from experience, but are intuitions *a priori*", he wrote in his 1890 *Introduction to the Theory of Knowledge or Philosophy* ("Einleitung in die Wissenslehre oder Philosophie"). "As long as the child has not learned to move its eyes and hands, it knows nothing of space; it only experiences space by its movements. And human beings also have no aprioristic knowledge that space consists of three extensions; rather, they can assume any given number of extensions; but external space has only three extensions, and this, again, man knows only from experience."[92]

Along with Lobačevskij[93], the Graßmann brothers were among the first mathematicians who, equipped with well-founded scientific knowledge, broke through the limits which Kant had set up for mathematics with his aprioristic concept of space.

4.3 Hermann Graßmann's views on restructuring mathematics and on locating *Extension Theory*

In his philosophical reflections on determining the different mathematical disciplines, Hermann Graßmann assumed that thinking is the activity of uniting, opposing and equating, of conjoining and separating: a dialectical process. By constructively defining the object of mathematics as the "particular as established by means of thought", he obtained a double conceptual and dialectical characterization – on the one hand "simply the particular", on the other, "coming to be". This conceptual development then took place – continuing his father's approach from the *Pure Theory of Number* (ZL) – within the dialectical opposition of discreteness and continuousness on the one hand, and of particularity and generality (as equality and inequality) on the other.

His account begins by developing the modes of becoming of the mathematical forms: "Each particular existent brought to be by thought ... can come about in one of two ways, either through a simple act of *generation* or through a twofold act of *placement and conjunction*. That arising in the first way is the *continuous form*, or *magnitude* in the narrow sense, while that arising in the second way is the *discrete* or *conjunctive form*."[94] If, then, the discrete magnitude is a "mere conjunctive act"[95] of the given, Graßmann already viewed the process of "generating" a continuous form as a complicated dialectical act. In analogy to the act of generating a discrete form, "one discerns in the *concept* of the continuous form a twofold act of placement and conjunction, but in this case the two are united in a single act, and thus proceed together as an indivisible unit."[96]

Here Graßmann was saying that the dialectical process of movement, of becoming (which he explicated by showing the movement of mathematical reflection), in its conceptual components, always brings us to oppositions (placement and conjunction) which, in turn, must be grasped in their unity: "Both acts, placement and conjunction, are thus merged together so that conjunction cannot precede placement, nor is placement possible before conjunction. …that which newly emerges does so precisely upon that which has already become, and thus, in that moment of becoming itself, appears in its further course as growing there."[97]

From this point of view, becoming is a contradiction which constantly resolves itself, only to reappear again.

Even though mathematics, in this sense, is organized in continuous and discrete components, these opposing domains are not completely isolated: "The opposition between the discrete and the continuous is (as with all true oppositions) fluid, since the discrete can be regarded as continuous, and the continuous as discrete. The discrete may be regarded as continuous if that conjoined is itself again regarded as given, and the act of conjoining as a moment of becoming. And the continuous can be regarded as discrete if every moment of becoming is regarded as a mere conjunctive act, and that so conjoined as given for the conjunction."[98]

The second opposition, which demanded a close examination of mathematical disciplines, arose from an analysis of the concept of the particular: "Each particular existent becomes such through the concept of the *different*, whereby it is coordinated with other particular existents, and through this with the *equal*, whereby it is subordinated to the same universals with other existents. That arising from the equal we may call the *algebraic form*, that from the different the *combinatorial form*."[99]

"From the interaction of these two oppositions[100] [continuous – discrete, equal – unequal; H.-J. P.], the former of which is related to the type of generation, the latter to the elements of generation, arise four species of form and the corresponding branches of the theory of forms"[101].

Up to this point, Hermann Graßmann was essentially following his father's ideas.[102] But the subsequent elaboration of the dialectical schema:

	discrete	**continuous**
equal		
unequal		

went beyond the framework of his father's developments.

While Hermann Graßmann and his father shared the view that the "algebraic-discrete form" represented the object of the theory of number and that the "combinatorial-discrete form" was the object of the theory of combinations, theoretical differences appeared when it came to determining the mathematical discipline arising from the "continuous synthesis of equals" – that is to say, the discipline whose object is the "algebraic-continuous form". According to Justus Graßmann, this discipline was geometry.

But Hermann Graßmann – after having banned geometry from the domain of pure mathematics – was closer to the point when he connected the "algebraic-continuous form" to the "intensive magnitude": when variable magnitudes "constitute the foundation of function theory, that is differential and integral calculus"[103].

The "combinatorial-continuous form" (unequal and continuous), for which Justus Graßmann had not been able to find a place in the structure of mathematics, had now become essential:

	discrete	**continuous**
equal	theory of number	geometry
unequal	combinatorics	?

According to Hermann Graßmann, the equivalent of this form was the "extensive magnitude", which was the basis for his *Extension Theory*. "It is thus somewhat as if the intensive magnitude is number become fluid," he explained, "the extensive magnitude [that is, vectors and multivectors – H.-J. P.] combination become fluid."[104]

He thereby obtained the following complete classification of the basic mathematical disciplines:

	discrete	**continuous**
equal	theory of number	theory of intensive magnitudes (theory of functions, differential and integral calculus)
unequal	combinatorics	theory of extensive magnitudes (extension theory)

These developments are very enlightening when it comes to understanding the genesis of Graßmann's theory of *n*-dimensional manifolds.

In Graßmann's way of thinking, which was strictly dialectical, his father's incomplete dialectical schema must have been unacceptable. In its incompleteness, it must therefore have exerted a heuristic pressure upon him. After having excluded geometry from the domain of pure mathematics, his *Extension Theory*, though at first limited to three dimensions as the theory of extensive magnitudes in a Kantian sense[105], could only position itself in a blank space in his father's schema as a "continuous combination"[106]. In this analogy between the theory of combinations and *n*-chain terms of "complexions" (permutations), a generalization to *n* dimensions had become a conceivable step.

It is likely that this approach, arising from a dialectical mindset, was one of the roots of Graßmann's generalized concept of geometry.

Enriques' assertion: "The possibility of a … generalization [of the concept of geometry to *n*-dimensions – H.-J. P.] had been hinted at, for example, by Herbart, whose philosophical views, as we all know, exerted a strong influence on Graßmann's and Riemann's way of thinking"[107], which we can also find in the *History of Science*[108], cannot be upheld with such absolute certainty.

Unquestionable historical proof exists for Herbart's influence on Riemann.[109] But Herbart's name appears in none of Graßmann's writings accessible to us. The only thing we know is that a brother-in-law of Graßmann, the headmaster of the Stettin Friedrich-Wilhelmsschule C. G. Scheibert, was a follower of Herbart's philosophical and pedagogical teachings.[110] We cannot exclude the *possibility* of Herbart's ideas influencing Graßmann through this connection, but there is no proof to justify the rigorousness in the quotation above. Interestingly, Roberto Torretti has pointed out that Riemann's construction of an *n*-dimensional manifold has more in common with Graßmann's ideas than with Herbart's. He therefore concludes that Riemann possibly was inspired by Graßmann.[111]

If one wanted to accept Enriques' claim, then one could also say that Graßmann was under the sway of Kant's early writings when he found the concept of *n*-dimensional space. In Kant's treatise *Thoughts on the True Estimation of Living Forces* ("Gedanken von der wahren Schätzung der lebendigen Kräfte", Kant 1929) from 1746, we find the following interesting thoughts in the first section:

Riemann, Herbart and the concept of *n*-dimensional manifolds

Riemann presented his conception of *n*-dimensional manifolds in his habilitation lecture *On the Hypotheses which lie at the Bases of Geometry* (Riemann 1876a). In this lecture, he pointed directly to inspiration he had received from Herbart.[112] In Riemann's philosophical fragments (Riemann 1876b) he further remarked: "The author is a Herbartian in psychology and epistemology…, but he usually cannot accept … Herbart's philosophy of nature and the related metaphysical disciplines."[113]
It seems probable that Herbart's elaborations on "conceptive series of lower and higher order" in his *Psychology*[114] were mostly responsible for Riemann's creation of the concept of *n*-dimensional manifolds.

Here we read:
"The interconnection of series consists in the fact that, when several of them are realized, not only each term simultaneously creates a series (that is, a conceptive series – H.-J. P.) starting from it, but also that the secondary series are united according to a rule with other series, term by term; so that the points where they join are always more than one, and that the construction *curves back into itself* in infinite multiplicity, without creating an internal contradiction with itself. The product of such series, which reciprocally generate one another, is in any case a *spatial one*, though not necessarily one in the sensible space of the world."[115]

According to Kant, the three-dimensional extension of space depends on the effect of the forces caused by the substances.[116] Therefore he concludes "that from a different law an extension with other properties and dimensions would have arisen. A science of all these possible kinds of space would undoubtedly be the highest enterprise which a finite understanding could undertake in the field of geometry."[117] The young Kant's remarkable reflections prove that the "critical Kant" was more dogmatic about his concept of space than the "pre-critical Kant". Graßmann could certainly have related to some of the points made by Kant.

Nevertheless, we shall return to the thoughts Graßmann developed in his *Extension Theory*.

In the following words, he once more summed up his object of investigation: "As with number there prevails the unification of that imagined together, and with combination the separation; so also with the intensive magnitude there appears the unification of elements, indeed separate conceptually, but which form the intensive magnitude only in their essential quality. In contrast, with the extensive magnitude there prevails the separation of elements that are indeed unified insofar as they form a single magnitude, but which constitute that magnitude precisely in their mutual separation."[118]

Summing up, we should note in this context that Graßmann decomposed the constructive limits of mathematical objects into their opposing components. He then isolated these oppositions and separately integrated them into the foundations of the corresponding mathematical disciplines, with one side of the opposition elevated to form the point of departure of the subsequent reflections, while the corresponding other side of the opposition is always kept in mind.[119]

This is how Graßmann moved from dialectical thought to "metaphysical" determinations, giving mathematics the shape of a dialectically developed "metaphysics".[120]

But this view of mathematics as a dialectically unfolded "metaphysics" is already contradictory in a dialectical sense and carries with it at least three aspects in which the non-dialectical contents must be put into perspective:

Mathematical reflection is essentially movement and, like every movement, dialectical in nature.

The basic terminology and basic objects of mathematics, even if they have arisen from the act of isolating the opposing sides of a dialectical unity, are in themselves another complicated dialectical unity, a lower level of dialectical complexity and a potential for development within the corresponding mathematical discipline. In this sense, every formal mathematical construction contains as much objective dialectics as the concepts and their definitions manage to capture.[121]

This means that through combining basic terminology, through analysis, synthesis and construction, new concepts, objects and relationships can be created – or necessarily come to be –, which again necessarily have dialectical properties (the more differential power and complexity a mathematical construction possesses, the stronger its dialectical potential in mathematical thinking will be).

But at the same time Graßmann was also aware of the objective dialectics inherent to the phenomena of nature and knew that it was possible to grasp various relationships in reality with mathematics. "It is also clear", he remarked, "that each real magnitude can be viewed in two ways, both as intensive and extensive: thus lines may also be regarded as intensive magnitudes if one removes from their nature the way their elements lie apart, and retains simply the quantity of their elements; and in the same way a point associated with a specific force can be thought of as an extensive magnitude, since one can represent the force in the form of a line."[122]

Even though Graßmann had managed to more or less *isolate* the mathematical disciplines by his (historically limited) classification, and had thereby attempted to obtain greater clarity in his concepts and subject of study, he went another important step further by conceiving the general theory of forms: "Antecedent to the division of the theory of forms into its four branches is a more general subject that we may call the general theory of forms. In it are presented the general conjunctive laws that apply to all branches alike."[123]

This "general theory of forms" – a term also used by Hankel at a later point in time[124] – was supposed to provide the basis for all of mathematics. Graßmann's way of proceeding shows a profoundly dialectical mindset, along with the goal of reaching the highest possible level of abstraction in order to highlight the universal within the particular. In the process of developing his theory of extensive magnitudes, Graßmann had provoked first radical changes in mathematics, and this led him to submit the whole of mathematics to a close analysis. This radical change, which potentially had already been a part of mathematical developments up to this point in time and which Graßmann had created in a specific way – by transferring terms, rules and approaches from arithmetic to geometry ("negative" displacements, "addition" of displacements, "multiplication" of displacements etc.) –, involved at least a threefold epistemological and methodological analysis:

a) The analysis of the new theory's position in the total structure of mathematics;
b) The confirmation, or revision, of the worldview and philosophy making up the foundation of this new mathematical structure;
c) The analysis of the invariance of terms and operations with respect to all mathematical disciplines, or the generalization of those principles and concepts which have a universal meaning in all fields of mathematics.

While we have already discussed Graßmann's treatment of the points a) and b) in detail, point c) must be placed in the context of his attempt to develop a "general theory of forms".

By applying mathematical language to objects for which, strictly speaking, it was no longer valid – see above – and for which mathematical language could merely provide a pattern of thought, yielding new concepts and relationships (the non-commutativity of vector multiplication as opposed to "common" multiplication, for example), the attention of scientists was now drawn to the invariant elements in *every* mathematical conjunction. As a consequence, the structure of a certain number of formal conjunctions had become visible, conjunctions whose specificities were then elaborated in different disciplines, the objects of which had arisen from their own mathematical synthesis. While, from a philosophical point of view, Graßmann's point of departure had been the conceptual fragmentation of the old unity into a multiplicity of mathematical disciplines, he also searched for the remaining structural unity within this multiplicity. He then placed this unity, as an *intra-mathematical* foundation, before the whole structure of mathematics.[125]

In developing his general theory of forms, Graßmann, who in the spirit of Schleiermacher flatly opposed an axiomatic approach to mathematics, once again remained true to himself. As L. G. Birjukova and B. V. Birjukov pointed out, the *Extension Theory* of 1844 "presented, for the first time in the history of science, a concise genetic and constructive program for the structure of mathematics. In the process, *Extension Theory* also formulated a system of postulates which yielded a large number of modern algebraic structures [semigroups, groups, rings, etc. – H.-J. P.]...".[126]

Thus, obviously, the general theory of forms contains no principles about what might be valid in mathematics, nor does it include axioms concerning content or form – which later led to frequent creative misreadings, as in Whitehead's formal foundation of mathematics[127]. Instead, the general theory of forms remains within the framework of Graßmann's constructive approach, which assumes that the objects of mathematics arise from their own, particular synthesis. On the one hand, it grows upon the universal preconditions of mathematical synthesis. On the other, it stands for the anticipated universal element in the concrete manifestations of mathematical synthesis.[128] The latter legitimate the former. Whatever in the further course of *Extension Theory* appears as an interpretation serves as a heuristics for the reader and is, in Graßmann's sense, a return from the abstract to the concrete.[129]

4.4 Hermann Graßmann's views on the essence of the mathematical method and its relation to the method of philosophy

Building on the analysis of his creative work and linking himself to his father's ideas, Hermann Graßmann, under Schleiermacher's direct influence, came to essential insights on the relation between the mathematical and philosophical methods. Even though just the fact that these insights became part of his work in individual scientific fields is exemplary, Graßmann's effort to explicitly formulate these insights, thereby attempting to produce more than merely "pure" knowledge[130], but also to provide the methodological tools for attaining and further elaborating this knowledge, is all the more worthy of our admiration. Graßmann's way of proceeding shows how strong and sustainable dialectical thinking can be. His point of departure was the following: "The philosophical method characteristically proceeds by oppositions, and thus progresses from the general to the particular, whereas the mathematical method proceeds from the most elementary concepts to the more complex, and thus produces new and more general concepts by the conjunctions of particulars."[131]

Here Graßmann clearly echoed the Kantian view on the relation between the mathematical and philosophical methods.[132] But he went beyond the Kantian position – inspired by Schleiermacher[133] – by characterizing the philosophical method as a dialectical one. Furthermore, in contrast to Kant, Graßmann shifted the focus of his concept of science by letting himself be influenced by the spirit of the Romantic philosophy of nature: "Thus whereas in the former [in philosophy – H.-J. P.] the overview over the whole predominates, and its development consists precisely in the gradual ramification and articulation of the whole, in the latter the interconnection of particulars is emphasized, and separate, independent developments combine together, each becoming only a factor in the following concatenation. This difference in method is implicit in their concepts; for in philosophy the primitive is precisely the unity of the idea, the particular being derived, whereas in mathematics the particular is the primitive, the unifying idea, the last aspired to; and thus are caused their opposing developments."[134]

By comparing the philosophical to the mathematical method, Graßmann concluded that both had to have something in common. On the one hand, he found this common feature in their demands for scientific rigor, on the other, in the fact that both require a mode of presentation favoring an overview of the whole.[135]

For Graßmann, the issue of the overview – resulting from his study of the geometrical methods of Euclid and Descartes – was extraordinarily important because he felt that the reader had to stay on top of theoretical developments if he wanted to gain independent insights. Only then would the reader be able to master the material himself and become independent from "that particular method by which he found that truth established."[136]

Graßmann linked this form of presentation to the concept of the idea: "For each given part of the presentation, the nature of its further development is essentially fixed by a dominant idea that is either nothing more than a supposed analogy with cognate and already familiar branches of knowledge, or, and this is the better case, is a direct intuition of the next succeeding truth."[137]

A heuristic pattern (analogy) and intuition (presentiment) hereby become the keys to the act of mathematical creation. Even though Graßmann showed little interest in analogy, calling it a "makeshift", the developmental history of *Extension Theory* proves him wrong and indirectly confirms a statement of Poincaré's, who noted that "…despite the exceptions … it is none the less true that sensible intuition is in mathematics the most usual instrument of invention."[138] But Graßmann thought highly of intuition. He emphasized that one cannot find new truths by "blind combinations"[139]. The leading idea was to be that "an overall view of the entire course of development leads to a new truth, but not yet at an opportune moment for its development, and thus initially only as a dim presentiment; the detailing of that moment includes both the discovery of the truth and the critique of the presentiment."[140]

Other great scientists later repeatedly emphasized the importance of scientific intuition, which Graßmann described so fittingly here. But Graßmann did not content himself with merely witnessing this phenomenon, but drew direct conclusions on how science should present its findings: "The essence of a scientific presentation is thus an interlocked pair of approaches, one of which leads consistently from one truth to another and forms the actual content, while the other controls the treatment and thus determines the form."[141]

But since in mathematics "these two separate lines of development diverge most acutely"[142], it is important for its form of presentation that "the reader be placed in the position in which a discoverer of the truths is certain to be found *in the best cases* [italics added – H.-J. P.]. In these circumstances, as the truths are discovered there occurs a continual reflection about the course of development; a characteristic line of thought develops in the reader about the procedure followed and about the idea lying at the foundation of the whole; and this line of thought forms the true nucleus and spirit of his activity, while the consistent detailing of the truths is only the embodiment of that idea."[143]

In his *Extension Theory*, Graßmann did his best to communicate the historical, psychological and logical aspects of his concept of science to the reader. He unfolded – as far as the analysis of his own approach would let him – his entire program of inquiry, his "paradigm": the ruptures and elements of continuity in his new theory, embedded in a foundational philosophical framework.

This ambitious project was one of the reasons why, on the one hand, his scientific results were ignored by his mathematical contemporaries, and, on the other, once his theory had become valued in mathematics, why an entire mathematical school – the "Graßmannians" – emerged from it.[144]

4.5 Graßmann's *Extension Theory* and Schleiermacher's *Dialectic*

The scientific literature about the history of mathematics has repeatedly posed the question whether the form of the *Extension Theory* of 1844 may be linked to a certain philosophical movement or, more precisely, whether an individual philosopher might be responsible for it. Without a doubt, we can find indirect connections to Leibniz, Schelling and Kant. These were influences in which Justus Graßmann also played a part. When it comes to direct influences, things are more complicated. Herbart has been named[145], but a direct influence on his part is highly improbable and therefore should be ruled out. Judging from the evidence we have, Fries – who also has been mentioned – is also an unlikely candidate.[146]

We have also rejected the unconvincing position that Graßmann experienced no direct influence from any philosopher at all, but only from his younger brother Robert.[147]

As the only remaining alternative, the present biography brings forward the view that when Hermann Graßmann expressed utmost respect for Schleiermacher and his exemplary philosophical and methodological approach to science in the curriculum vitae of 1833, submitted to the examination committee in theology, he was sincere[148] about his feelings. Therefore we can assume that his further scientific work was substantially influenced by Schleiermacher.

A. R. Schweitzer (1915), in his dissertation[149], was probably the first to favor this point of view, saying that Schleiermacher's influence on Graßmann was especially strong in his reflections on the problem of what "guiding idea" may be behind the evolution of mathematics. Then this line of thought was abandoned and only taken up again later by Lewis (1977) and, independently, by the author (1979a). Lewis begins his text by discussing Schleiermacher's *Dialectic* separately, and then continues with Graßmann's *Extension Theory*. He concludes that Schleiermacher exerted a major influence on Graßmann, but does not make this connection completely clear. This led to Schubring's (1996a) rejection of Lewis' arguments for, in his view, being inconsistent. Schubring denies the possibility of any relevant influence of Schleiermacher on Graßmann. Instead, relying on new documentation, he thinks it is highly probable that Robert Graßmann stimulated his brother philosophically. Even though Lewis defended some of the points he made in 1977 in an essay published in 2004, he generally – as the essay's title suggests – reduced the force of his initial argument.

The following paragraphs aim to substantiate the view that not only Lewis was right, but that Schleiermacher's influence was even greater than Lewis assumed in 1977.

The foundation of *Extension Theory*, the internal development of its concepts and its external integration into the totality of the sciences, is a closed, unified structure and neatly fits into the conception of Schleiermacher's *Dialectic*. A detailed synopsis would lead too far. Instead, the following paragraphs will assemble evidence for the Graßmann brothers' profound respect for Schleiermacher's "philosophical theory of scientific discovery", or "Erspähungslehre" in the words of Robert Graßmann[150]. They will attempt to paraphrase the basic idea of *Extension Theory* in the spirit of Schleiermacher's *Dialectic*. In this context, the foundation of *Extension Theory* reads as follows (the corresponding paragraphs from the German Jonas edition of the 1839 *Dialectic* appear in parentheses. Positions attributable to Graßmann are set in italics in the following text):

The architecture of knowledge relies on the positive double relative opposition (§§ 289/290; §§ 340/341). This opposition has its ultimate ground in the opposition of the ideal and the real in reason and nature, an opposition whose contradictory moments reappear – in shifting relations of dominance – on both of its sides (§ 341).

The highest division of the sciences into real and formal sciences (or, in Schleiermacher's words, into empirical and speculative sciences) follows this principle (§ 342). These sciences, then, internally split up again into an opposition of the real and the formal. This means that the formal sciences are structured into a science dealing with the general (dominated by speculative thinking) and a science dealing with the particular (dominated by empirics): the latter is mathematics, the former dialectics (§§ 344 – 346). *But the opposition between the ideal and the real remains alive in mathematics: So mathematics again splits up into a predominantly speculative part (the general theory of forms) and a predominantly empirical part (the four mathematical disciplines, generated by the double relative opposition between equal and unequal, continuous and discrete). On the other hand, dialectics also possesses a predominantly empirical part, apart from the mostly speculative part. The former is formal logic and is related to mathematics.*

According to Schleiermacher, all knowledge is thought (§§ 86sqq), that is, a process in which thought refers to an existent (§§ 9sqq) and in which truth relies on the correspondence of thought to this existent. If this is the case, knowledge in mathematics, as a formal science standing next to the empirical sciences (§ 346), relies on a thought *that refers to an existent posited by thought itself. It possesses truth in the correspondence of thought to this existent*. Since all thought is also a part of the opposition between the real and the ideal, it is always a unity of the intellectual and organic functions, of concept and schema (§§ 105sqq). *The particular existent, which mathematics posits as its object, is devoid of all content, a pure thought-form of the real, with the highest possible degree of ideality and a minimum of intuition: a conceptual schematism, open to any possible content taken from reality.* Since all knowledge arises from an act of production (§§ 87sqq), *only the characteristics of this productive act of positing the particular can form the basis of mathematics. According to Graßmann, this particular may only be posited conceptually as equal or unequal, discrete or continuous.* These four concepts are the concepts creating a positive double relative opposition, which fulfills all of Schleiermacher's requirements for the foundation of an architecture of science (§§ 289sqq). *The general theory of forms deals with the analysis of the conditions under which the possibility of mathematical synthesis and the conjunctions* *it posits arise, as required by this opposition.* This analysis gives us the necessary general dimension of any given conjunction (principle of congruence – § 331) and therefore not only serves a basis of proof, but also as a heuristics for finding conjunctions in the different branches of mathematics. The "principle of analogy" (§ 332) gives us a second element of this heuristics. *In his* Extension Theory*, Graßmann obtains this principle from geometry.* In the best case, it is accompanied by presentiment as the intuition of further possibilities of developing a mathematical construction (§ 330). We can hereby determine the nature of the true development of a mathematical theory. It is the interlacing of two opposing methods (§§ 240ff.), namely of a heuristic and predominantly philosophical method, with an architectonical method, characterized by the rigor of mathematical construction.

In this sense, Graßmann's *Extension Theory* of 1844 can be read as the strict realization of the principles of Schleiermacher's *Dialectic*. In the process, Graßmann extends them to form a "dialectic of mathematics", compensating for the inadequacies in Schleiermacher's treatment of mathematics.[151]

In this context, the pair of oppositions equal/unequal and discrete/continuous, as developed by Justus Graßmann, constitutes the philosophical and mathematical point of departure for the project of harmoniously embedding mathematics into Schleiermacher's dialectical program, a project superseding the Kantian point of view.

But how did Justus Graßmann find this approach?

It is a tempting option to suppose that Justus Graßmann's conception of mathematics had already arisen under Schleiermacher's influence, and that Justus and Hermann Graßmann's view followed the Schleiermacherian path.

Justus Graßmann's years as a student in Halle from 1799 to 1801 coincided with the heyday of the philosophy of nature. Shortly afterwards, in 1804, Schleiermacher came to Halle and gave lectures in which he developed an early version of his philosophical ethics. He became a close friend of Henrik Steffens, who had become a professor for the philosophy of nature. Notes taken in Schleiermacher's first lectures began to circulate among interested readers. Bartholdy, a representative of the local school board in Stettin who was advocating a better education for elementary school teachers and the foundation of a pedagogical seminar,[152] had also received a copy. An intimate friend of Schleiermacher and also of Justus Graßmann, Bartholdy studied this text carefully with Gaß, copied it down for himself and spoke very positively about Schleiermacher's ideas. According to a letter from Gaß to Schleiermacher from July 1805, the lectures on philosophical ethics contained a detailed explanation of Schleiermacher's transcendental postulates. Here he developed his own position as an alternative to Schelling's. Gaß wrote: "Bartholdy is especially appreciative of your deviation from Schelling, whose first lecture on academic studies we also read, and wishes that in this matter you may never move in his direction again."[153]

This opposition to Schelling consisted in, as Arndt concluded, the view that the contrast of the ideal and the real in the domain of objective knowledge was a relative one,[154] and we therefore also have good reason to assume that Justus Graßmann came into contact with Schleiermacher's conception of dialectics as early as 1805.

Since we also know that Gaß possessed a copy of Schleiermacher's 1814/15 *Dialectic*[155] (Schleiermacher had sent the booklet to Breslau for Gaß to study it, as he had already done with the *First Draft of Ethics),* we must also consider the possibility that Bartholdy, who had met Schleiermacher rather frequently because he was a correspondent and member of the Berlin scientific deputation (after 1810), was also well informed about Schleiermacher's *Dialectic*. Even though Bartholdy died in May 1815, it is probable that Justus Graßmann, who continued Bartholdy's work with the 1817 *Geometry for Elementary Schools*[156], knew a lot about the essential ideas of Schleiermacher's *Dialectic* and was inspired by them.

Unfortunately, we are speculating now and more hard proof is unlikely to appear.

It is important to know that the fascinating mathematical and philosophical edifice of the *Extension Theory* of 1844 could not withstand the process of its further mathematical elaboration, which according to Robert Graßmann set in as early as 1847.[157] After only a few years – though almost unnoticed – it collapsed.

The *Textbook of Arithmetic* of 1861 meant the end for the underlying concept of a double opposition which was supposed to provide the foundation for mathematics. *The idea behind the whole conceptual structure, including the idea of erecting a general theory of forms, had become obsolete.*

Giuseppe Veronese: ***On the Foundations of Geometry of Multiple Dimensions...***

Taking essential passages from Graßmann's introduction to his *Extension Theory* of 1844 as a point of departure in his foreword[158], Veronese opens his treatise with the following paragraphs:

"§ 1. I think.
§ 2. I think **one thing** or **multiple things**. ...
§ 3. I think one thing first, after which I think one thing. ...
§ 4. *Def.* That phenomenon which **corresponds** to a thing in thought, we will call the *representation*, *concept* or *mental image* of a thing.
§ 5. *Des.* A single or multiple things or concepts are designated by signs, e. g. with letters of the alphabet. ...
§ 6. *Def.* When I think a thing, one will say: The thing has been *given* or *posited* by the thought; when I think of a thing, one will say: The thing is a *given* for the thought."[159]

He continues with an analysis of the fundamental operations, and goes on to develop the first "properties of abstract mathematical forms". Then he develops arithmetic and the elementary systems of one dimension, and finally arrives at an analysis of the continuum and the forms of multiple dimensions.

Now, in a grandiose construction, *all of mathematics had become included in extension theory*. But the philosophical foundational structure it had once possessed had been lost in the course of this development. Mostly because Graßmann faced a total lack of appreciation for his mathematical and, especially, his mathematical and philosophical achievements in the scientific community, he never undertook the project of recreating this foundational structure.

Nevertheless we might surmise that Giuseppe Veronese's 1891 (philosophical) foundation for a geometry of multiple dimensions – the German translation of his work appeared exactly 50 years after Graßmann's first *Extension Theory* in 1894 – might have come close to Hermann Graßmann's new concept of mathematics, if one leaves the heuristic method out of the question.[160]

Closing remarks

It was the intention of the present book to provide insights into the life and works of Hermann Graßmann, a scholar who today is unjustly underestimated and forgotten.

Fig. 47. Etching of Graßmann from the first volume of his collected works

It has become clear that Graßmann, rooted in the provinciality of his – not always beloved – job as a teacher in Stettin, was peculiarly capable of connecting himself to the *Zeitgeist* and to the fundamental problems of the scientific fields he was working in. Far ahead of contemporary scientific thought, but neglected and misunderstood, Graßmann made great progress. His historical merit lies in the way in which he partially abandoned old ways of thinking, consciously drawing inspiration from the revolutionary philosophical spirit of his time and foreshadowing developments in present-day mathematics.

Graßmann is worthy of entering into the ranks of the 19th century's most exceptional scientists. In our day, nobody can deny the importance of his scientific achievements, which for decades he had tried to communicate to his contemporaries.

The history of science has largely confirmed the hopeful words Graßmann wrote in 1861 in the foreword to his second *Extension Theory* and which will bring the present book to a close:

> "For I have every confidence that the effort I have applied to the science reported upon here ... is not to be lost. Indeed I well know that the form I have given the science is, and *must* be, imperfect. But I also know and must declare, even at the risk of sounding presumptuous, – I know, that even if this work as well should lie idle yet another seventeen years or more without influencing the living development of science, a time will come when it will be drawn forth from the dust of oblivion and the ideas laid down here will bear fruit."[161]

Notes

1 Leibniz, G.W.: *Project of a New Encyclopedia to be written following the method of invention* (15 June 1679). In: Dascal 2008, p. 130 – 141 (p. 133 – 134).

2 A1, p. 9.

3 See chapter 1.

4 A1, p. 9.

5 We possess no further information about this little text. – See BIO, p. 36sq.

6 See chapter 2, section 1.

7 See chapter 1.

8 A1, p. 9. See also Justus Graßmann's teaching manual of plane and spherical geometry (1835), which deals with "The positive and negative in geometry" in § 59 (p. 26sqq) and with "The positive and negative in space" in § 163 (p. 70sqq), while also showing that geometry is the specific realm of negative magnitudes and that arithmetic only possesses this concept to the extent to which it is connected to geometry.

9 A1, p. 9.
10 See J. Graßmann 1824, p. 194. – Hermann Graßmann directly refers to this passage from his father's treatise in the foreword to his *Extension Theory.* – See A1, p. 9.
11 A1, p. 9.
12 Ibidem.
13 Ibidem.
14 A1, p. 10.
15 Bell 1986, p. 360.
16 See ibidem.
17 In his *Pure Theory of Number* ("reine Zahlenlehre", ZL), Justus Graßmann explicitly pointed out that the conjunction of base and exponent, known as a power, is not commutative. – Details on this in chapter 2, section 1.
18 A1, p. 10.
19 Ibidem.
20 See chapter 1, section 4.
21 See Graßmann's letter to the minister of culture Eichhorn, cited in chapter 1, section 6, which shows clearly that it was only in 1840 that Graßmann grasped the far-reaching consequences of his vector algebraic investigations.
22 Poincaré 2007, p. 79/80.
23 See chapter 3, section 2.
24 A1, p. 10.
25 KRY, p. 174.
26 See chapter 3, section 3.
27 Möbius 1827, p. xiv.
28 Möbius 1827, p. 16/17.
29 A1, p. 11.
30 See chapter 2, section 1.
31 A1, p. 12.
32 See chapter 3, section 3.
33 A1, p. 17.
34 See chapter 1, section 5, for a selection of the many unfavorable judgments important mathematicians pronounced on the *Extension Theory's* philosophical tone.
35 A1, p. 15.
36 A1, p. 16.
37 See Klaus 1965, p. 102.
38 A1, p. 23. An alternate translation of this passage by Albert Lewis renders "produced by thought alone" as "posited through thought itself".
39 See A1, p. 23sq.

40 For the religious dimensions in Justus Graßmann's mathematical writings, see chapter 2, section 1.
41 See Ruzavin 1977, p. 172.
42 See chapter 2, section 3.
43 DIAL, p. 7.
44 DIAL, p. 48.
45 DIAL, p. 50.
46 DIAL, p. 53.
47 DIAL, p. 40 (note).
48 DIAL, p. 48 (note).
49 Hermann Graßmann's curriculum vitae of 1833; quoted from BIO, p. 21.
50 A1, p. 23. Where Lloyd Kannenberg uses the concept of portrayal, we would like to suggest representation as an alternative.
51 H. Graßmann 1878, p. 7.
52 Ibidem, p. 9/10.
53 Ibidem, p. 15.
54 A1, p. 23/24.
55 A1, p. 24.
56 DIAL, p. 309.
57 For Schleiermacher's views, see chapter 2, section 3.
58 The reason for this is connected to the fact that Graßmann followed Schleiermacher's concept of dialectics, which was mainly a "structural and functional dialectics".
59 Concerning the Hegelian philosophy's bad reputation among the natural scientists of the 19th century, see Friedrich Engels: The Old Preface to Anti-Dühring. On Dialectics (1878). MECW, vol. 25, p. 336 – 344.
60 A1, p. 23.
61 Ibidem.
62 Ibidem, p. 24.
63 Ibidem.
64 See section 3 of the preceding chapter.
65 See Kant 2007, p. 577sq.
66 See Kant 2007, p. 65sqq.
67 See Šljachin 1976, p. 131.
68 Kant 2007, p. 577.
69 See Šljachin 1976, p. 131.
70 ZL, p. 3.
71 Ibidem.
72 Ibidem.
73 ZL, p. 3.

74 Ibidem, p. 4.
75 Schleiermacher 1942, p. 130.
76 DIAL, p. 386 (footnote).
77 Schleiermacher 1942, p. 33.
78 Einstein is very explicit in a remark in *Geometry and Experience*: "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality." See Einstein 2004, p. 12 – 23.
79 A1, p. 23/24. Again, in this quotation we would like to suggest "posited" as an alternate translation for "established".
80 See Kant 2007, p. 578sq.
81 A1, p. 24.
82 See Molodschi 1977, p. 210sq.
83 In 1860 and the following years, the Graßmann brothers restored the designation 'theory of magnitude' for mathematics, while expelling the theory of combinations to the domain of logic.
84 Bourbaki 1950, p. 231.
85 A1, p. 46.
86 See also Molodschi 1977, p. 276/277.
87 R. Graßmann 1890b, p. 70.
88 A1, p. 11.
89 See chapter 2, section 1.
90 A1, p. 24.
91 A1, p. 24/25.
92 R. Graßmann 1890b, p. 70.
93 See, for details on Lobačevskij's philosophical positions, Licis 1976, p. 52sqq.
94 A1, p. 25.
95 A1, p. 26.
96 A1, p. 25.
97 Ibidem. "Positing and connecting" might very well serve as alternate translations for "placement and conjunction" in these quotations.
98 A1, p. 26.
99 Ibidem.
100 We should note that the concept of the interaction of two crossing oppositions is a basic idea of Schleiermacher.
101 A1, p. 26.
102 For Justus Graßmann's division of mathematics into different disciplines, see chapter 2, section 1.
103 A1, p. 27.
104 Ibidem, p. 27.

105 See Kant 2007, p. 197sqq.
106 ZL, p. 6.
107 Enriques 1907, p. 63/64.
108 History of science 1965, p. 34sq.
109 One might compare Herbart's views to Riemann's elaborations, cited in chapter 3, section 3, referring to the generation of *n*-dimensional manifolds.
110 More information on Scheibert's relation to Herbart's philosophy can be found in Müller 1926.
111 See Torretti 1984, p. 107 – 109. We find a contrary argument in Laugwitz 1999, p. 223.
112 See Riemann 1876a, p. 255.
113 Riemann 1876b, p. 476.
114 Herbart 1890, p. 409sqq.
115 Ibidem, p. 415.
116 See Kant 1929, p. 12.
117 Ibidem.
118 A1, p. 27.
119 Compare Graßmann's realization: "…although the different always somehow adheres to the equal and conversely, at the moment of consideration only the one is encompassed, while the other only appears as the requisite basis of the first." (A1, p. 26).
120 Here the concept "metaphysical" is meant to signify "un-dialectical", not "anti-dialectical". See Heitsch 1976, p. 56. While the dialectical dimension prevails in the A1 of 1844, the "metaphysical" approach is dominant in the A2 of 1862.
121 See Erpenbeck/Hörz 1977, p. 133.
122 A1, p. 28.
123 Ibidem.
124 See chapter 3, section 7.
125 See A1, p. 28.
126 Birjukova/Birjukov 1997, p. 137.
127 See chapter 3, section 7.
128 See chapter 3, section 3.
129 See also the detailed analysis in Radu 2000, p. 173sqq. Birjukova/Birjukov 1997 also come very close.
130 Herewith Graßmann stands for a positive alternative to the "prince of mathematics", Gauß, who almost never uncovered his scientific work-process. See Wußing 1976, p. 64sq.
131 A1, p. 30.
132 See Kant 2007, p. 577sq.
133 See chapter 2, section 3 for more details on Schleiermacher's philosophical views.
134 A1, p. 30.
135 See A1, p. 30.

136 A1, p. 31.
137 Ibidem.
138 Poincaré 2007, p. 25.
139 A1, p. 31.
140 Ibidem.
141 A1, p. 32.
142 Ibidem.
143 Ibidem.
144 In this context, see the statistical analyses of works published by Graßmann's followers on the one hand, and Hamilton's on the other, ranging from the second half of the 19th century to the early 20th century. In: Crowe 1994, p. 109 – 149.
145 See chapter 4, section 3.
146 See chapter 2, section 2.
147 Also chapter 2, section 2.
148 Given Hermann Graßmann's straightforward character, it seems highly unlikely that he was merely using his curriculum vitae to improve his chances of success in a theological exam.
149 There is a reference to Schweitzer in Lewis 2004. But when Lewis points out that Paul Carus, who was a pupil of Graßmann in Stettin, also was strongly familiar with Graßmann's way of thinking, this is only partly true, because Carus strongly opposed the mathematical concept of *n*-dimensional spaces as late as the 1880s (Carus 1881): "But the whole enterprise has no other significance than to show that what in fact is impossible may still be correct from a logical point of view. It is a scientific equivalent to a paradoxical piece of fiction and only confirms the undeniable fact that one cannot contradict the careful liar by his own words alone, because one will find no contradiction there." (Carus 1881, p. 54) Carus changed his mind 25 years later (Carus 1908), when he published a foundation of a philosophy of geometry, in which he praised Graßmann enthusiastically. So Paul Carus and Victor Schlegel seem to share the same attitude. Both Carus and Schlegel admired Graßmann's ideas only after having left Stettin.
150 See chapter 2.2.
151 See also R. Graßmann 1890b, p. 83, and chapter 2, section 2.
152 Schleiermacher wrote Gaß about Bartholdy: "In Berlin, he told me about his plans for a seminar, which made me very happy, and from which I conclude that I share his view of Pestalozzi's idea and its essential importance. Here, as well, the combination of Lutheran and Reformed higher schools is the next step." Letter from Schleiermacher to Gaß, May 1805. In: Schleiermacher 1852, p. 23.
153 Letter from J. Chr. Gaß to Schleiermacher, 13 July 1805. In: Schleiermacher 1852, p. 25sq.
154 See Arndt 1986, p. xxiii.

155 "With much gratitude I return your copy of the *Dialectic* …, my dear friend." Letter from J. Chr. Gaß to Schleiermacher, 31 March 1816. In: Schleiermacher 1852, p. 125sq.

156 Justus Graßmann wrote in the preface to the *Geometry for Elementary Schools* (1817): "Enough has already been said about the justification and usefulness of this work in the foreword to Bartholdy's 'Versuch einer Sprachbildungslehre für Deutsche', which has appeared recently. Its general outlines have something to do with some local schools for the poor, which I and some dear friends, now passed away, most notably councilor Bartholdy, have helped to found and establish, apart from preparing these school's teachers for the sake of doing a good deed, voluntarily and without pay." (Graßmann 1817, p. iii).

157 See R. Graßmann 1891, p. 16sq.

158 See Veronese 1894, p. vii sqq.

159 Veronese 1894, p. 1 – 2.

160 See also Radu 2000, p. 166.

161 A2, p. xvii.

Chronology of Graßmann's life

1809 15 April: Hermann Günther Graßmann is born in Stettin. Napoleonic troops have been occupying the city since November 1806.

1814 Hermann Graßmann begins to attend a private school. He will attend the Stettin "Gymnasium" for 7 ½ years.

1817 Justus Graßmann publishes *Geometry for Elementary Schools, Part 1* (J. Graßmann 1817).

1820 In autumn the ballad composer Carl Loewe comes to Stettin and joins the Graßmanns in their apartment for a year.

1824 Graßmann's father publishes *Geometry for Elementary Schools, Part 2* (J. Graßmann 1824).

1827 Justus Graßmann publishes *On the Concept and Extent of the Pure Theory of Number* (ZL).

1827 Möbius publishes *Barycentric Calculus* (Möbius 1827).

1827 17 September: Hermann Graßmann graduates from "Gymnasium". He and his brother Gustav begin their studies of theology at the University of Berlin.

1829 Justus Graßmann publishes *Physical Crystallonomy and the Geometrical Theory of Combinations* (KRY).

1830 In autumn Hermann Graßmann ends his studies in Berlin and returns to Stettin. Until December 1831 he works on mathematics, physics, mineralogy and botany.

1831 At Easter Graßmann obtains a position at the Stettin Seminar for Teachers.

1831 In December Graßmann privately works on the treatise *On Geometrical Analysis and the Application of Arithmetic and Algebra to Geometry*.

1831 On 17 December Hermann Graßmann passes the first examination for teachers.

1832 Hermann Graßmann works on botany and mathematics. First steps in vector algebra. Studies Plato's *Dialogues*.

1833/34 First examination in theology. He signs up for the examination on 23 March 1833, completes the written assignment, passes the oral examination (21 May 1834) and receives his graduation certificate on 22 May 1834.

1834 Graßmann obtains a position at the Berlin School of Commerce on 29 September.

1835 Hermann Graßmann makes corrections in his father's *Textbook of Trigonometry* (J. Graßmann 1835).

1835 In December Graßmann gives up his position at the Berlin School of Commerce and returns to Stettin on 1 January 1836. He will never leave Stettin again.

1836 Graßmann becomes a teacher at the "Ottoschule".

1838 On 24 May Graßmann signs up for the second examination in theology.

1839 On 28 February Graßmann asks the examination commission in Berlin for a second round of examinations in physics and mathematics.

1839 On 10 March Graßmann receives the written assignment on "low and high tides".

1839 Between May and June Graßmann takes the second examination in theology.

1839 Schleiermacher's *Dialectic* (DIAL) is published posthumously.

1840 On 20 April Graßmann sends the examination thesis on low and high tides to the commission in Berlin. On 1 May he takes the oral examination in mathematics, physics, mineralogy and chemistry.

1840 Hermann and Robert Graßmann study Schleiermacher's *Dialectic* (DIAL).

1842 At Easter Hermann Graßmann returns to ideas from his examination thesis and begins to present "extension theory" to a small family circle.

1843 At Easter Graßmann obtains a newly created position at the "Friedrich-Wilhelmsschule".

1843 In autumn Graßmann finishes work on the first volume of *Extension Theory* (A1).

1843 On 16 October Hamilton discovers quaternions. That same year, Möbius publishes his *Celestial Mechanics* ("Mechanik des Himmels", Möbius 1887a) in Germany.

1844 Graßmann visits Möbius in Leipzig.

1844 Graßmann's *Linear Extension Theory* (A1) is in bookstores.

1845 Hermann Graßmann publishes his treatise *New Theory of Electrodynamics* and has to review *Extension Theory* himself (H. Graßmann 1845a, 1845b). That same year he publishes an article on a *Purely Geometrical Theory of Curves*. Eleven more articles will follow.

1845 On 2 February Möbius informs Graßmann in a letter about the prize question issued to commemorate Leibniz's 200th birthday.

1845 Graßmann reads Schleiermacher's *Aesthetic* and spends the following year studying Hegelian philosophy with his brother.

1846 Graßmann publishes a *Collection of Texts for Young Readers from Age 8 to 12* with Langbein (Graßmann/Langbein 1868).

1846 On 1 July Graßmann is awarded the prize honoring Leibniz for his *Geometrical Analysis* (PREIS). Ideas from his *Extension Theory* receive public recognition for the first time.

1846 In autumn Graßmann begins to revise traditional mathematical concepts with his brother.

1847 In May Graßmann for the first time attempts to receive a professorship in mathematics at a university. Kummer's assessment destroys Graßmann's hopes.

1847 The "hunger revolts" prompt Hermann and Robert Graßmann to focus on politics. The two brothers read Schleiermacher's *Theory of the State* (1840) and Dahlmann's *Politics* (1835).

1848 On 18 and 19 March the bourgeois Revolution shakes Germany.

1848 On 1 April Graßmann publishes his first political article.

1848 Hermann and Robert Graßmann publish the conservative "German Weekly for Politics, Religion and National Life" from 20 May to 24 June. On 1 July the journal is replaced by the daily newspaper "North-German Newspaper".

1848 Hermann Graßmann and Marie Therese Knappe become engaged.

1849 Hermann Graßmann and Marie Therese Knappe get married on 12 April.

1849 Graßmann publishes his last article in the "North-German Newspaper" on 27 September. In February 1850 he leaves the newspaper while his brother carries on. Graßmann resumes his work on mathematics and physics and begins to study Sanskrit.

1852 Graßmann's father dies on 9 March. In July Graßmann becomes his father's successor at the Stettin "Gymnasium". A month later he receives the title of Professor.

1852 On 28 October Graßmann gives a lecture in the Stettin Physics Society: "On the newest discoveries in the field of color theory". He presents his color laws for the first time. In December he replaces his father as the head of the Physics Society.

1853 Graßmann resumes his correspondence with Möbius. He publishes an article on color theory.

1853 Hamilton's *Lectures on Quaternions* (Hamilton 1853) are published. Hamilton admiringly mentions Graßmann.

1854 Graßmann's article *Sur les différents genres de multiplication* is dated 5 February. The article defends his rights of authorship against Cauchy. In April Graßmann addresses the Academy in Paris, claiming priority for his discoveries. The Academy does not respond.

1854 On 10 June Riemann gives his habilitation lecture *On the hypotheses which lie at the Bases of Geometry* (Riemann 1876a).

1854 On 2 November Graßmann gives a lecture on the theory of vowels in the Physics Society.

1855 to 1856 Hermann and Robert Graßmann work on a new foundation for number theory, extension theory, logic and combinatorics.

1855 Graßmann publishes five articles on the theory of curves.

1856 Graßmann becomes a speaker at the Stettin lodge of Freemasons.

1857 Graßmann becomes a member of the Society for the Evangelization of China. At Easter 1858 he begins to publish the Society's leaflet "Reports from China" (1858 – 61).

1860 Graßmann publishes his *Textbook of Arithmetic* (LA). That same year, he publishes his first article on philology.

1861 The completely revised and restructured version of *Extension Theory* (A2) is published.

1862 For the second time, Hermann Graßmann officially applies for a professorship in mathematics. When the project fails, he feels disappointed with mathematics and focuses exclusively on Sanskrit and *Rig-Veda*.

1862 The philological article in which Graßmann develops his law of aspiration is dated 4 September (H. Graßmann 1863).

1864 On Knoblauch's initiative Graßmann becomes a member of the Society for Natural Science Research in Halle.

1865 Publication of Graßmann's *Textbook of Trigonometry* (H. Graßmann 1865).

1866 Graßmann begins an exchange of letters with Hankel in which the two discuss certain aspects of *Extension Theory*.

1867 Publication of Hankel's *Theory of Complex Number Systems* (Hankel 1867). In the book, Hankel recognizes the importance of Graßmann's work.

1868 Graßmann believes that thanks to Grunert he has a chance of obtaining a professorship in mathematics. By March of the following year, he realizes that his hopes have been in vain.

1869 In January Klein discovers Graßmann by reading Hankel.

1869 At Easter Graßmann's oldest son sets out to study mathematics in Göttingen. He brings Clebsch and Stern a copy of *Extension Theory*.

1869 Schlegel begins to work on his *System of Geometry. According to the Principles of Graßmann's Extension Theory and an Introduction Thereto* (Schlegel 1872a).

1870 Graßmann and Fuess discuss a heliostat designed by Graßmann. The heliostat is built in 1872.
1870 Hermann Graßmann publishes *German Plant Names* (H. Graßmann 1870).
1871 On 2 February Clebsch begins his correspondence with Graßmann.
1871 In autumn Klein studies Graßmann's *Extension Theory* while elaborating his Erlangen Program (Klein 1974). He informs Lie of Graßmann's treatment of the Pfaffian problem.
1871 On Clebsch's initiative Graßmann becomes a member of the Göttingen Science Society on 2 December. During the same reunion Clebsch gives a commemorative lecture for Plücker (Clebsch 1871) in which he repeatedly mentions Graßmann's achievements. Reassured by these signs of recognition, Graßmann resumes some of his mathematical work.
1872 After years of silence, Graßmann publishes some articles on mathematics. In December he completes his *Rig-Veda* dictionary (H. Graßmann 1955). The process of editing the book extends until 1875.
1872 Graßmann attends the Reunion of Philologists in Leipzig.
1872 In late autumn Schlegel's *System of Geometry* is published (Schlegel 1872a).
1872 Lie visits Graßmann in Stettin in October and November to learn about Graßmann's approach to the Pfaffian problem.
1872 Clebsch dies on 7 November.
1874/75 Graßmann's health begins to deteriorate.
1875 In August Graßmann is diagnosed with a valvular heart defect.
1876 Preyer visits Graßmann in Stettin on 11 March to discuss his theory of sensations.
1876 Graßmann is made a member of the American Oriental Society. On Roth's initiative, the University of Tübingen accords Graßmann an honorary doctorate for his philological work.
1877 Graßmann writes and publishes a large number of treatises in mathematics and physics.
1877 Graßmann has to give up teaching on 27 August for health reasons.
1877 Hermann Günther Graßmann dies on 26 September.
1878 Graßmann's last treatise *On the Loss of Faith* (H. Graßmann 1878) is published. Also, a second printing of Graßmann's *Extension Theory* of 1844 is in bookstores.
1894 to 1911 Publication of Graßmann's collected works.

A list of Graßmann's writings can be found in (GW32).

Abbreviations

GW11 Graßmann, H.: Gesammelte mathematische und physikalische Werke. Bd. 1.1. Herausgeg. von Fr. Engel unter Mitwirkung von E. Study. Leipzig 1894. (Reprint: New York 1969 and 1972).

GW12 Graßmann, H.: Gesammelte mathematische und physikalische Werke. Vol. 1.2. Herausgeg. von Fr. Engel unter Mitwirkung von H. Graßmann (d. J.). Leipzig 1896. (Reprint: New York 1972).

GW21 Graßmann, H.: Gesammelte mathematische und physikalische Werke. Vol. 2.1. Herausgeg. von E. Study, G. Scheffers und Fr. Engel. Leipzig 1904. (Reprint: New York 1972).

GW22 Graßmann, H.: Gesammelte mathematische und physikalische Werke. Vol. 2.2. Herausgeg. von J. Lüroth und Fr. Engel. Leipzig 1902. (Reprint: New York 1972).

GW31 Graßmann, H.: Gesammelte mathematische und physikalische Werke. Vol. 3.1. Herausgeg. von J. Graßmann und Fr. Engel. Leipzig 1911. (Reprint: New York 1972).

GW32 Graßmann, H.: Gesammelte mathematische und physikalische Werke. Vol. 3.2. Herausgeg. von Fr. Engel. Leipzig 1911. (Reprint: New York 1972).

BIO Engel, Fr.: Graßmanns Leben. Nebst einem Verzeichnisse der von Graßmann veröffentlichten Schriften und einer Übersicht des handschriftlichen Nachlasses. In: (GW32, p. 1 – 400).

ENCYK Encyklopädie der mathematischen Wissenschaften mit Einschluß ihrer Anwendungen. Vol. 1 to 6. Herausgeg. im Auftrage der Akademien der Wissenschaften zu Göttingen, Leipzig, München und Wien sowie unter Mitwirkung zahlreicher Fachgenossen. Leipzig 1898sqq.

A1 Graßmann, H. (1844): Linear Extension Theory. In: A new branch of mathematics. The *Ausdehnungslehre* of 1844 and Other Works by Hermann Graßmann. Translated by Lloyd C. Kannenberg. Chicago and La Salle: Open Court 1995, p. 3 – 312. Also in: (GW11, p. 4 – 312).

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List of illustrations

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